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CONSTRUCTION OF BALANCED INCOMPLETE BLOCK DESIGNS WITH UNEQUAL BLOCK SIZES

BY PP 44

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DEDICATION

This project is dedicated to the mystical body of Christ.

CERTIFICATION

This project report has not been in substance for any degree and is not being concurrently submitted for another degree.

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ABSTRACT

There are three major concepts of balancing in incomplete block designs viz variance balancing, pairwise balancing and efficiency balancing. The necessary and sufficient conditions for a general class of connected designs to be balanced is derived. It will be seen that for an incomplete block design with t treatments in b blocks of unequal sizes to be balanced, the inequality $b \geq t$ should hold.

The expression for calculating efficiency balanced incomplete block designs is shown.

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CHAPTER ONE

1.1 INTRODUCTION

> "Since statisticians do not usually perform experiment, their claim to attention when they write on experimentation requires some explanations", William Cochran and Gertrude Cox (1957).

Block designs are widely used in many fields of research. Their most common type is the randomized (complete) block design. It is not applicable in all circumstances. If the numbers of treatments is too large to preserve homogeneous conditions within complete blocks or the sizes of blocks are determined by the nature of the experimental material, then incomplete block designs are used. A wide range of these designs is available for planning experiments in blocks of equal sizes but with a smaller number of plots than the total number of treatments. An experimenter who wants all the contrasts between treatment to be confounded with blocks to the same extent may use a balanced incomplete block design (BIBD). Designs of that type can be constructed for any number of equally replicated treatments and any size of equal blocks, but may require quite a large number of replicates. If the experimenter is willing to accept that some of the possible treatment contrasts are more confounded than others, he may choose one of the various partially balanced incomplete block (PBIB) designs listed in standard works on experimental design, Cochran and Cox (1957). These designs require a smaller number of replications.

The high degree of symmetry in the pattern of BIB is in many respects a desirable property. But the restriction of the design to equal block sizes and equal number of treatment replications may be a serious practical obstacle in many experimental circumstances that naturally contain unequal block sizes. For instance if;

(a) Many new varieties or strains of a crop together with one or two standard varieties are to be compared in one field experiment. The number of plots for the new varieties is severely limited by the available amount of seed, but there is no such limitation for the standards.

(b) A wide range of virus inoculations are to be compared on a number of plant leaves, but the plants, chosen as blocks, have only small and varying numbers of leaves that may be taken as plots of the experiment.

(c) In a nutritional experiment the numbers of animals in the available litters are unequal, being in some cases equal to the number of treatments but in others smaller or larger.

This work will concentrate mainly on blocks of unequal sizes with equal or unequal replications of the treatments. It has been shown Rao (1958) that a necessary and sufficient condition for a design to be balanced [that is var($\hat{t}_i$-$\hat{t}_{ii}{}^1$) to be the same for all pairs (i,i^1), $i \neq i^1$], is that the matrix C of the adjusted intrablock normal equation shall have all its off - diagonal elements equal and all its diagonal elements equal. If r_i denote the number of times that the ith treatment is replicated, k_j denotes the number of plots in the jth block and $\mathbf{N} = (n_{ij})$ is the incidence matrix, then the design with $\mathbf{t}$ treatments in $\mathbf{b}$ blocks will be denoted by the parameters

$$\mathbf{t},\ \mathbf{b},\ \mathbf{r}_i\ (i = 1, 2, \ldots, t),\ \mathbf{k}_j\ (j = 1, 2, \ldots, \mathbf{b}),\ \lambda_{ii}{}^1$$

where

λ_{ii^1} is the number of blocks in which the pair ith and i^1th treatments ($i \neq i^1$) occur together.

1.2 NOTATIONS AND DEFINITION OF TERMS

For easier understanding and comprehension of this work the notations used will be explained and some terms will be defined. Matrices should be denoted by bold uppercase letters, for instance **R** is a diagonal matrix of treatment replications. Vectors will be denoted by bold small letters with a bar below them, for instance $\underline{\mathbf{r}}$ is a vector of treatment replications. Scalars will be denoted by small letters for instance, n is the total number of experimental plots.

N is the incidence matrix and is sometimes referred to as the design.

n is the total number of experimental units.

C is the matrix of adjusted intrablock normal equation.

$\boldsymbol{\Omega}$ is the pseudo variance - covariance matrix.

I is an identity matrix

$\underline{\mathbf{1}}$ is a vector of unit elements

$\underline{\mathbf{1}}\,\underline{\mathbf{1}}^1$ is a matrix with all elements as unity.

μ is an eigen value

$\mathbf{K}$ is the diagonal matrix of block sizes.

$\mathbf{X}$ (x_{uij}) is the plot x block matrix

$\underset{\sim}{\mathbf{T}}$ is a vector of treatment totals

$\underset{\sim}{\mathbf{B}}$ is a vector of block totals.

$\hat{\underset{\sim}{\alpha}}$ is the intrablock estimator

$\underset{\sim}{\mathbf{K}}$ is the vector of block sizes

The design is proper if $\mathbf{K}_j = \mathbf{K}$ for all j, equireplicate if $\mathbf{r}_i = r$ for all i and binary if n_{ij} takes only the values **0** or **1**.

A design is connected if and only if the rank of $C = t - 1$.

CHAPTER TWO

CONCEPTS OF BALANCING IN BLOCK DESIGNS

2.0 INTRODUCTION TO THE CONCEPTS

There are three different concepts of balancing in Incomplete Block Designs namely;

i) Pairwise balance

ii) Variance balance

iii) Efficiency balance.

Given the equation

$$\Omega^{-1} = \mathbf{R} - \mathbf{N}\mathbf{K}^{-1}\mathbf{N}^{1} + \underline{\mathbf{r}}\,\underline{\mathbf{r}}^{1}/n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.0.1)$$

See e.g. Tocher (1952) where $N = (n_{ij})$ is the t x b incidence matrix with a row for each treatment and a column for each block.

$$n = \sum_{i=1}^{t} r_i = \sum_{j=1}^{b} k_j$$

equation (2.0.1) is used extensively in various aspects of balanced designs.

2.1 PAIRWISE BALANCED BLOCK DESIGN (PBBD)

In a pairwise balanced block design, Hedayat and Stufken (1989), any two treatments form a pair within the blocks equally often. If all block sizes are equal, a class of such designs is formed by the balanced block designs (BBD's), of which the well known balanced incomplete block designs (BIBD's) are a subclass.

The concept of pairwise balanced design was introduced by Bose and Shrikhande (1960). It was later generalized by Hedayat and Federer (1974) who defined a block design as pairwise balanced if all the off diagonal elements of

$$\mathbf{N}\,\mathbf{N}^{1} = \mathbf{D} + \theta\,\underline{\mathbf{1}}\,\underline{\mathbf{1}}^{1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.1.1)$$

where $\mathbf{D}$ is a $t \times t$ diagonal matrix and θ is a scalar constant.

Example 2.1.1

Consider an equireplicate design with an unequal block sizes given by the parameters

$$t = 6,\ b = 22,\ r = 8,\ k_j = \begin{cases} 3 \text{ for } j=1, \ldots, 4 \\ 2 \text{ for } j=5, \ldots, 22, \end{cases}$$

$\lambda = 2$.

The blocks are

1.	(1, 2, 3)	12.	(2, 4)
2.	(2, 3, 4)	13.	(2, 5)
3.	(4, 5, 6)	14.	(2, 5)
4.	(1, 5, 6)	15.	(2, 6)
5.	(4, 5)	16.	(2, 6)
6.	(1, 2)	17.	(3, 4)
7.	(1, 3)	18.	(3, 5)
8.	(1, 5)	19.	(3, 5)
9.	(1, 4)	20.	(3, 6)
10.	(1, 4)	21.	(3, 6)
11.	(1, 6)	22.	(4, 6)

$$N N^1 = \begin{bmatrix} 8 & 2 & 2 & 2 & 2 & 2 \\ 2 & 8 & 2 & 2 & 2 & 2 \\ 2 & 2 & 8 & 2 & 2 & 2 \\ 2 & 2 & 2 & 8 & 2 & 2 \\ 2 & 2 & 2 & 2 & 8 & 2 \\ 2 & 2 & 2 & 2 & 2 & 8 \end{bmatrix}$$

$$= 6I + 2\underline{1}\,\underline{1}^1$$

hence $D = 6I$, $\theta = 2$ but the C - matrix associated with the design is

$$
\mathbf{C} = \begin{bmatrix}
{}^{13}/_{3} & {}^{-5}/_{6} & {}^{-5}/_{6} & -1 & {}^{-5}/_{6} & {}^{-5}/_{6} \\
{}^{-5}/_{6} & {}^{13}/_{3} & {}^{-2}/_{3} & {}^{-5}/_{6} & -1 & -1 \\
{}^{-5}/_{6} & {}^{-2}/_{3} & {}^{13}/_{3} & {}^{-5}/_{6} & -1 & -1 \\
-1 & {}^{-5}/_{6} & {}^{-5}/_{6} & {}^{13}/_{3} & {}^{-5}/_{6} & {}^{-5}/_{6} \\
{}^{-5}/_{6} & -1 & -1 & {}^{-5}/_{6} & {}^{13}/_{3} & {}^{-2}/_{3} \\
{}^{-5}/_{6} & -1 & -1 & {}^{-5}/_{6} & {}^{-2}/_{3} & {}^{13}/_{3}
\end{bmatrix}
$$

Since $N\,N'$ is in the form of $\mathbf{D} + \theta \underline{1}\ \underline{1}'$ the design is said to be pairwise balanced. However, the design is not variance balanced because the off - diagonal elements of the C - matrix are not all equal.

2.2 VARIANCE BALANCED DESIGN (VBD)

A block design is said to be variance balanced if every normalized estimable linear function of treatment effects can be estimated with the same precision.

THEOREM 2.2.1

The necessary and sufficient condition for a design to be variance balanced is that the **C** - matrix

$$\mathbf{C} = \mathbf{R} - \mathbf{N}\,\mathbf{K}^{-1}\,\mathbf{N}' \qquad \text{(2.2.1)}$$

has all its diagonal elements equal and all its off-diagonal elements equal.

Proof

Ref - [Rao (1958); Theorem 1]

Hedayat and Stufken (1989) defined a design to be variance balanced (for treatment effects), if the information matrix for treatment effects is a completely symmetric matrix, that is if it is of the form

$$aI + b\underline{1}\,\underline{1}' \text{ for constants a and b.}$$

Under this design, the least squares estimate of elementary treatment contrasts all have the same variance.

In view of the fact that

$$\mathbf{C}\,\underline{\mathbf{1}} = \underline{\mathbf{0}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.2.2)$$

where $\underline{\mathbf{0}}$ is a $t \times 1$ null vector, the equality of the off diagonal elements of **C** (as well as **P**) where

$$\mathbf{P} = \mathbf{N}\,\mathbf{K}^{-1}\,\mathbf{N}' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.2.3)$$

implies that all the diagonal elements of **C** must be equal. A necessary and sufficient condition for a block design to be variance balanced is that **P** - matrix has all its off diagonal elements as the same value and it can be expressed as

$$\mathbf{P} = \mathbf{D} + \theta\,\underline{\mathbf{1}}\,\underline{\mathbf{1}}' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.2.4)$$

2.3 EFFICIENCY BALANCED DESIGN

A block design is said to be efficiency balanced if all treatment contrasts $\underset{\sim}{S}^1\underset{\sim}{T}$.

Satisfy $M_o \underset{\sim}{S} = \mu \underset{\sim}{S}$ .... (2.3.1)

where μ is a unique non-zero eigen value of M_o with multiplicity t - 1 and

$$M_o = R^{-1}P - \underset{\sim}{1}r^1/n$$

In such a design, every treatment contrast is estimated with the same relative efficiency.

THEOREM 2.3.1 [Puri & Nigam (1977)]

Given $\underset{\sim}{r}^1 = (r_1, r_2, \ldots, r_t)$

$\underset{\sim}{K}^1 = (K_1, K_2, \ldots, K_b)$

$R = \text{diag}\,(r_1, r_2, \ldots, r_t)$

$K = \text{diag}\,(K_1, K_2, \ldots, K_b)$

a necessary and sufficient condition for a connected block design to be efficiency balanced is that

$$P = \mu R + (1 - \mu)\, \underset{\sim}{r}\, r^1/n$$

The efficiency factor of the design is $1 - \mu$.

CHAPTER THREE

BALANCED INCOMPLETE BLOCK DESIGNS

3.1 FACTORIAL DESCRIPTION OF INCOMPLETE BLOCK DESIGN

Given a factorial experiment with p factors $A_1, A_2, \ldots, A_p$ where the kth factor has a_k levels and a treatment represents a level or combination of levels of experimental factors, then for the P - factor experiment, the total number of treatments is given by

$$t = a_1 \, a_2 \, a_3 \ldots ap = \prod_{k=1}^{P} a_k \quad \ldots\ldots \qquad (3.1.1)$$

Conventionally, we code the levels of the kth factor by 0, 1, 2, ..., a_k-1 where each treatment is a P - tuple $t_i = x_{1i} \, x_{2i} \ldots\ldots x_{pi}$; $i = 1, 2, \ldots, t$

$$x_{ki} = 0, 1, 2, \ldots, a_k - 1;\ k = 1, 2, \ldots, p \quad \ldots\ldots\ldots\ldots \qquad (3.1.2)$$

We can as well give each of the t treatments an integer value,

$$1, 2, \ldots, t \text{ hence } t_i = i \text{ for } i = 1, 2, \ldots, t \quad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.1.3)$$

If the factorial experiment is arranged in **b** blocks where the jth block is of size $K_j \leq t$ and the ith treatment is observed a total of r_i times in the design, then the known intra block model may be defined by

$$Y_{ij} = \mu + \propto_i + \beta_j + e_{ij} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.4)$$

$(i = 1, 2, \ldots t; j = 1, 2, \ldots, b)$.

The reduced normal equation for estimating the treatment effects having adjusted for blocks is

$$Q = C\hat{\propto} \ldots\ldots\ldots\ldots \quad (3.1.5)$$

where

$$\mathbf{Q} = \underline{\mathbf{T}} - \mathbf{NK}^{-1}\underline{\mathbf{B}} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.6)$$

$$\mathbf{C} = \mathbf{R} - \mathbf{NK}^{-1}\mathbf{N}^{1} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.7)$$

where $\underline{\mathbf{T}}$ and $\underline{\mathbf{B}}$ are as defined in section 1.2, $\hat{\propto}$ is the intra block estimator of $\propto$ (the treatment effects). $\hat{\propto} = (\hat{\propto}_1, \hat{\propto}_2, \ldots, \hat{\propto}_t) \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.8)$

Onukogu (1995) has shown that

$$K = X^1X = \text{diag}(k_1, k_2, \ldots, k_b) \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.9)$$

where $x = (x_{uj})$ is the plot x block matrix;

$$X_{uj} = \begin{cases} 1 & \text{if the treatment belongs to the jth block} \\ 0 & \text{otherwise.} \end{cases}$$

hence equation (3.1.7) becomes $\mathbf{C} = \mathbf{R} - \mathbf{N}(\mathbf{X}^1\mathbf{X})^{-1}\mathbf{N}^1 \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1.10)$

A design is said to be variance balanced if every elementary contrast $(\propto_i^{\,1})$ is estimated with the same variance.

It was shown by Rao (1958) that the necessary and sufficient condition for a design to be variance balanced is that **C** has (t-1) equal latent roots other than zero.

Recall from eqn. (3.1.10) that

$$C \quad = \quad \mathbf{N - NK^{-1} N^{1}}$$

with the condition that $\underline{\mathbf{1}}^{1}\ \hat{\alpha} \ = \ \mathbf{0}$ .... (3.1.11)

We know that if the rank $C = t - S$, a set of $S - 1$ independent contrasts are not estimable, but if the rank $C = t\text{-}1$, every contrast is estimated and in this case the design is said to be connected.

If the design is connected there are $t\text{-}1$ non zero latent roots, say $\lambda_1, \lambda_2, \ldots, \lambda_{t-1}$. According to Rao (1958), as the rows of C add to zero, $(t^{-\frac{1}{2}}, \ldots, t^{-\frac{1}{2}})$ is the latent vector corresponding to the root zero.

$$\text{Let } L = \begin{bmatrix} l_1 \\ \cdots\cdots \\ t^{-\frac{1}{2}}\ \underline{\mathbf{1}}^{1} \end{bmatrix} = \begin{bmatrix} (l_{ij}) \\ \cdots\cdots \\ t^{-\frac{1}{2}}\ \underline{\mathbf{1}}^{1} \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.1.12)$$

be an orthogonal matrix transforming C into diagonal form.

Since L is orthogonal

$$I = L^{1} L = L^{1} L_1 + {}^{1}/_{t}\ \underline{1}_{t}\ \underline{1}^{1}_{t} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.1.13)$$

pre multiplying (3.1.5) by L, we get

$$L\,Q = L\,C\ \hat{\alpha} = \begin{bmatrix} \text{diag}\ (\lambda_1, \lambda_2, \ldots, \lambda_{t-1}) \\ \cdots\cdots\cdots\cdots \\ \underline{0} \end{bmatrix} L_1\ \hat{\alpha} \quad \ldots. \qquad (3.1.14)$$

Hence we get

$$L_1\ \hat{\alpha} = \text{diag}\ ({}^{1}/\lambda, \ldots, {}^{1}/\lambda_{t-1})\ L_1\ Q$$

Pre-multiplying by L'_1 and using (3.1.11) and (3.1.13) we obtain

$$\hat{\alpha} = \mathbf{D}\,\mathbf{Q} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.1.15)$$

where

$$\mathbf{D} = [d_{ij}] = L'_1 \operatorname{dig}\,[{}^{1}/\lambda_1, \ldots, {}^{1}/\lambda_{t-1})\, L_1 \ \ldots\ldots\ldots \qquad (3.1.16)$$

From solution (3.1.15) it follows that $V(\hat{\alpha}) = D\,\sigma^2$

$$V(\hat{\alpha}_i - \hat{\alpha}_j) = (d_{ii} + d_{jj} - 2\,d_{ij})\,\sigma^2$$

$$= \sum_{r=1}^{t-1} \left(\frac{l_{ri} - l_{rj}}{\lambda r}\right) \sigma^2 \quad \ldots\ldots\ldots\ldots \qquad (3.1.17)$$

The average variance is

$$\frac{1}{t(t-1)} \sum_{i} \sum_{\substack{j=1 \\ i \neq j}}^{t} V(\alpha_i - \alpha_j)$$

$$\frac{2\sigma^2}{t-1} \sum_{r=i}^{t-1} \frac{1}{\lambda_r} \ldots\ldots\ldots\ldots \qquad (3.1.18)$$

Theorem 3.11 (Rao (1958))

A necessary and sufficient condition for a design to be variance balanced is that C has t-1 equal latent roots other than zero.

Proof

To prove that the condition is necessary it is enough to show that $\lambda_1 = \lambda_2 = \ldots\ldots =$

λ_{t-1}, for the **C** - matrix of the balanced design is of rank t-1.

From equation 3.1.17 and 3.1.18 above we get

$$\sum_{r=1}^{t-1} \frac{(l_{ri}-l_{rj})^2}{\lambda r} \frac{2}{t-1} \sum_{r=1}^{t} \lambda r$$
$$= \frac{1}{t-1}\sum_{k=1}^{t-1} \frac{1}{\lambda k}\sum_{r=1}^{t-1} (l_{ri}-l_{rj})^2 \quad ..... \tag{3.1.19}$$

hence

$$\sum_{r=1}^{t-1} (l_{ri}-l_{rj})^2 \left(\frac{1}{\lambda r}-\frac{1}{t-1} \sum_{k=1}^{t-1} \frac{1}{\lambda k}\right) = 0 \;........ \tag{3.1.20}$$

Consider

$$V(\hat{\alpha}_j - \hat{\alpha}_k) = V(\hat{\alpha}_i - \hat{\alpha}_j) + (\hat{\alpha}_i - \hat{\alpha}_k) -$$
$$2\ COV\ [(\hat{\alpha}_i - \hat{\alpha}_j), (\hat{\alpha}_i - \hat{\alpha}_k) \;................ \tag{3.1.21}$$

hence

$COV\ (\hat{\alpha}_i - \hat{\alpha}_j), (\hat{\alpha}_i - \hat{\alpha}_k) =$

$$\frac{1}{t-1} \sum_{r=1}^{t-1} \frac{1}{\lambda r} \quad .... \tag{3.1.22}$$

that is

Hence

From (3.1.20) and (3.1.24) taking i = 1, we get

.... (3.1.25)

$$\sum_{r=1}^{t-1} \frac{(l_{ri}-l_{rj})\ (lri-l_{rk})}{\lambda r} \tag{3.1.23}$$

$$= \frac{1}{t-1}\sum_{r=1}^{t-1} \frac{1}{\lambda r} \sum_{r^1=1}^{t-1} (l_{r^1i}-l_{r^1j})\ (lr^1j-l_{r^1k}) \ldots\ldots$$

$$\sum_{r=1}^{t-1} (l_{ri}-l_{rj})(l_{ri}-l_{rk})\ (\frac{1}{\lambda r}-\frac{1}{t-1}\sum_{r^1=1}^{t-1} \frac{1}{\lambda r}) = 0$$

$$i \neq j \neq k \ldots\ldots$$

$$d^{(j)^1} diag\ (\frac{1}{\lambda_1}-\frac{1}{t-1}\sum_{r=1} \frac{1}{\lambda r}, \ldots, \frac{1}{\lambda r-1} - \frac{1}{t-1}\sum_{r=1} \frac{1}{\lambda r}) d^k = 0$$

for j, k, = 2, 3, ..., t

where $d^{(j)}$ is the column vector $\{l_{11}-l_{1j},\ l_{21}-l_{2j},\ \ldots,\ l_{t-1,1}-l_{t-1,j}\}$

If

$$M = [d^{(2)}, d^{(3)}, \ldots, d^{(t)}] \ldots\ldots\ldots\ldots\ldots\ldots \tag{3.1.26}$$

then

$$M^1 M = \begin{bmatrix} 2 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & 1 \\ \vdots & \vdots & & \\ 1 & 1 & \ldots & 2 \end{bmatrix}$$

and det $M^1 M = t \neq 0$, hence $M^1 M$ and M are non singular.

Therefore $d^{(2)}, \ldots, d^{(t)}$ are t-1 linearly independent (t-1) vectors. Any (t-1) vector say ε, can be uniquely expressed in terms of these vectors, say

$$\varepsilon = C_2 d^{(2)} + C_3 d^{(3)} + \dots + C_t d^{(t)}$$

from (3.1.25) it follows that

$$\varepsilon^1 \, diag\left(\frac{1}{\lambda_1} - \frac{1}{t-1} \sum_{r=1} \frac{1}{\lambda r}, \dots, \frac{1}{\lambda_{t-1}} - \frac{1}{t-1} \sum_{r=1} \frac{1}{\lambda r}\right) \varepsilon = 0$$

$$= \sum_{i,j=2} C_i C_j \, d^{(i)^1} diag\left(\frac{1}{\lambda_1} - \frac{1}{t-1} \sum \frac{1}{\lambda r}, \dots, \frac{1}{\lambda_{r-1}} - \frac{1}{t-1} \sum_{r} \frac{1}{\lambda r}\right) d^{(j)} = 0 \dots\dots\dots \qquad (3.1.27)$$

By taking successively ε to be the (t-1) vectors (1, 0, 0, ..., 0), ..., (0, 0, ..., 1), we get

$$\frac{1}{\lambda_1} = \frac{1}{\lambda_2} = \dots\dots\dots\dots = \frac{1}{\lambda_t}$$

hence $\lambda = \lambda_1 = \dots = \lambda_{t-1}$

The condition is sufficient, for if rank C = t-1 and $\lambda_1 = \dots = \lambda_{t-1} = \lambda$ (say), it follows immediately that every elementary contrast is estimable, and the solution becomes

$$\hat{\alpha} = {}^1/_\lambda \, L_1^1 \, L_1 \, Q = {}^1/_\lambda \, (1 - {}^1/_t \, \underline{1}_t \, \underline{1}^1 t) \, Q = Q/\lambda \dots\dots\dots\dots\dots\dots\dots \qquad (3.1.28)$$

which shows that $V(\hat{\alpha}_i - \hat{\alpha}_j) = ({}^2/\lambda)\,\sigma^2 \dots\dots\dots\dots\dots\dots\dots \qquad (3.1.29)$

which is independent of both i and j hence the design is balanced.

QED

From the above, we find that (a) if the rank of the matrix C is exactly t-1, the design is said to be connected. This is a necessary and sufficient condition for all (elementary) contrasts of the treatment effects to be estimable. In this work, binary connected designs shall mostly be considered.

(b) We have seen that a design with t treatments is said to be variance balanced if all the elementary contrasts are estimated with the same precision, Raghavorao (1971). The necessary and sufficient condition for a connected design to be variance balanced are (Rao, (1958)).

i $\quad \mathbf{C}_{ii} = a \quad i = 1, 2, \ldots, t$ .... (3.1.36)

ii $\quad \mathbf{C}_{ii^1} = c \quad i \neq i^1 = 1, 2, t$ .... (3.1.37)

The elements of **C** are

$$C_{ii} = r_i - \sum_j (n_{ij}^2/k_j)\ldots\ldots \tag{3.1.38}$$

$$C_{ii^1} = \sum_j (n_{ij}\, n_{ij}^1 /k_j),\ i \neq j\ldots\ldots\ldots \tag{3.1.39}$$

<u>**Example 3.1.1**</u>

Consider an equireplicate proper design with the parameters

$t = 4,\ b = 6,\ r = 3,\ k = 2,\ \lambda = 1$

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

The C - matrix associated with the above design is

$$C = \begin{bmatrix} 5/3 & -1/3 & -1/3 & -1/3 & -1/3 & -1/3 \\ -1/3 & 5/3 & -1/3 & -1/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & 5/3 & -1/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & -1/3 & 5/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & -1/3 & -1/3 & 5/3 & -1/3 \\ -1/3 & -1/3 & -1/3 & -1/3 & -1/3 & 5/3 \end{bmatrix}$$

Since C_{ii} are equal $\forall i$ and $C_{ii}{}^{1}$ are equal $\forall i, i^1$, $(i \neq i^1)$ also $\mathbf{C\underline{1}} = \mathbf{O}$.

The design is balanced.

Example 3.1.2

Let us consider the example 4.2 given by Onukogu (1995) on a design with uniform blocks. The parameters are $\mathbf{t = A(3) \times B(3) \times (3) = 18, b = 3, k = 6}$.

The **C** - matrix associated with the above design is

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	5/6	0	0	-1/6	-1/6	0	0	0	0	0	0	-1/6	0	-1/6	-1/6	0	0	0
2	0	5/6	0	0	0	0	0	-1/6	-1/6	0	-1/6	0	-1/6	0	0	0	0	-1/6
3	0	0	5/6	0	0	-1/6	-1/6	0	0	-1/6	0	0	0	0	0	-1/6	-1/6	0
4	-1/6	0	0	5/6	-1/6	0	0	0	0	0	0	-1/6	0	-1/6	-1/6	0	0	0
5	-1/6	0	0	-1/6	5/6	0	0	0	0	0	0	-1/6	0	-1/6	-1/6	0	0	0
6	0	0	-1/6	0	0	5/6	-1/6	0	0	-1/6	0	0	0	0	0	-1/6	-1/6	0
7	0	0	-1/6	0	0	-1/6	5/6	0	0	-1/6	0	0	0	0	0	-1/6	-1/6	0
8	0	-1/6	0	0	0	0	0	5/6	-1/6	0	-1/6	0	-1/6	0	0	0	0	-1/6
9	0	-1/6	0	0	0	0	0	-1/6	5/6	0	-1/6	0	-1/6	0	0	0	0	-1/6
10	0	0	-1/6	0	0	-1/6	-1/6	0	0	5/6	0	0	0	0	0	-1/6	-1/6	0
11	0	-1/6	0	0	0	0	0	-1/6	-1/6	0	5/6	0	-1/6	0	0	0	0	-1/6
12	-1/6	0	0	-1/6	-1/6	0	0	0	0	0	0	5/6	0	-1/6	-1/6	0	0	0
13	0	-1/6	0	0	0	0	0	-1/6	-1/6	0	-1/6	0	5/6	0	0	0	0	-1/6
14	-1/6	0	0	-1/6	-1/6	0	0	0	0	0	0	-1/6	0	5/6	-1/6	0	0	0
15	-1/6	0	0	-1/6	-1/6	0	0	0	0	0	0	-1/6	0	-1/6	5/6	0	0	0
16	0	0	-1/6	0	0	-1/6	-1/6	0	0	-1/6	0	0	0	0	0	5/6	-1/6	0
17	0	0	-1/6	0	0	-1/6	-1/6	0	0	-1/6	0	0	0	0	0	-1/6	5/6	0
18	0	-1/6	0	0	0	0	0	-1/6	-1/6	0	-1/6	0	-1/6	0	0	0	0	5/6

C_{ii} are equal for all i but C_{ii^1} are not equal for all i, i^1 ($i \neq i^1$) hence the design is not variance balanced.

The balanced form of the above design will have the following parameters $t = A(3) \times B(3) \times C(2) = 18$, $b = 816$, $k = 3$, $r = 136$ $\lambda = 16$.

The **C** - matrix for the balanced design will be $96\ (I_{18} - {}^{1}/18\ \underline{1}_{18}\ \underline{1}'_{18})$

3.2 BALANCING AN INCOMPLETE BLOCK DESIGN WITH UNEQUAL BLOCK SIZES AND AN UNEQUAL REPLICATIONS OF EACH TREATMENT

If the condition of equal block sizes is relaxed, a variance balanced design need not be equireplicated. In this section, a method of constructing a class of variance balanced designs with unequal replications, unequal block sizes is presented. A $\lambda_{ii}{}^{1}$ being the number of blocks in which the pair of the ith and i^1th ($i \neq i^1$) treatments occur together.

Let us consider the usually BIB design with parameters

$$\left[t,\ b = \binom{t}{k},\ r = \binom{t-1}{k-1}, k, \lambda = \binom{t-2}{k-2}\right]$$

The variance balanced designs can be constructed; see, Khatri (1982) by taking any k different treatments out of t. Now let P be an integer such that $\mathbf{KP < t}$, and select P blocks out of blocks in such a way that no two blocks have any common treatments. Put these KP treatments of the P blocks into one block and repeat this block $\boldsymbol{\lambda_0 P}$ times

$$b = \binom{t}{k}$$

choose other blocks as the same blocks from

$$where \quad \lambda_o = \binom{KP-2}{k-2}$$

$$b = \binom{t}{k}$$

blocks after deleting

$$b_o = \binom{kp}{k}$$

blocks containing k treatments of KP chosen treatments. This will give the new designs with parameters

$t^* = t$, $b^* = b - b_0 + \lambda_0 P$,

$$K^*_j = \begin{cases} Kp \text{ for } j = 1, 2, \ldots, \lambda_0 P \\ = K \text{ otherwise} \end{cases}$$

$$r^*_i = \begin{cases} \lambda_0 P + r - r_0 \text{ for } i = 1, 2, \ldots, Kp \\ r \text{ otherwise} \end{cases}$$

$$where \quad r_o = \binom{kp-1}{k-1}$$

This design has the same C - matrix as C = qC_o. where q = t λ/k = b(k-1) / (t-1) and $C_o = I_t - t^{-1} \underline{1}_t \underline{1}'_t$

Example 3.2.1

Let us consider the class of BIB designs with t = 8 treatments in b = ½t(t-1) = 28 blocks of size 2 with r = (t-1) = 7 replications and

$$\lambda = \binom{t-2}{k-2} = 1$$

as the concurrence number.

The block contents are:

1.	(1, 2)	8.	(6, 7)	15.	(4, 7)	22.	(2, 6)
2.	(3, 4)	9.	(2, 7)	16.	(6, 8)	23.	(3, 5)
3.	(5, 6)	10.	(1, 4)	17.	(1, 6)	24.	(4, 8)
4.	(7, 8)	11.	(3, 6)	18.	(2, 4)	25.	(1, 8)
5.	(1, 3)	12.	(5, 8)	19.	(3, 8)	26.	(2, 5)
6.	(2, 8)	13.	(1, 5)	20.	(5, 7)	27.	(3, 7)
7.	(4, 5)	14.	(2, 3)	21.	(1, 7)	28.	(4, 6)

Let $P = 3$ since $2 \times 3 = 6 < 8$ we select $P = 3$ out of 28 blocks such that no two blocks have any common treatments and let the 2P treatments be 1, 2, 3, 4, 5 and 6. We put them in one block (1, 2, 3, 4, 5, 6) and we add this block $P = 3$ times.

Now we delete $P(2P-1) = 15$ blocks containing pairs of these $2P = 6$ treatments. The new incomplete block design with unequal block sizes are as follows:

1.	(1, 2, 3, 4, 5, 6)	7.	(2, 7)	12.	(3, 8)
2.	(1, 2, 3, 4, 5, 6)	8.	(5, 8)	13.	(1, 7)
3.	(1, 2, 3, 4, 5, 6)	9.	(4, 7)	14.	(4, 8)
4.	(7, 8)	10.	(6, 8)	15.	(1, 8)
5.	(2, 8)	11.	(5, 7)	16.	(3, 7)
6.	(6, 7)				

The parameters of this design are $t = 8$, $b = \frac{1}{2}\, t(t-1) - P(2P-1) + P = 16$

$$K_j = \begin{cases} 2P = 6 & \text{if } j = 1, 2, 3 \\ 2 & \text{otherwise} \end{cases}$$

$$r_j = \begin{cases} t - p = 5 & \text{if } i = 1, 2, \ldots, 6 \\ t - 1 = 7 & \text{otherwise} \end{cases}$$

$$\lambda_{ii^1} = \begin{cases} p = 3 & \text{if } i, i^1 = 1, 2, \ldots, 6 \ (i \neq i^1) \\ 1 & \text{otherwise} \end{cases}$$

The elements of C - matrix of this design will be

$C_{ii} \quad = \frac{1}{2}(t - 1) = {}^{7}/_{2} \; i = 1, 2, \ldots, 8$

$C_{ii}^{\;\;1} \quad = {}^{-1}/_{2} \;\; i, i^{1} = 1, 2, \ldots, 8 \; (i \neq i^{1})$

Example 3.2.2

Let the balanced incomplete block design have the following parameters

$(t = 7, b = 21, r = 6, k = 2, \lambda = 1)$

The blocks are

1.	(1, 2)	8.	(2, 4)	15.	(3, 7)
2.	(1, 3)	9.	(2, 5)	16.	(4, 5)
3.	(1, 4)	10.	(2, 6)	17.	(4, 6)
4.	(1, 5)	11.	(2, 7)	18.	(4, 7)
5.	(1, 6)	12.	(3, 4)	19.	(5, 6)
6.	(1, 7)	13.	(3, 5)	20.	(5, 7)
7.	(2, 3)	14.	(3, 6)	21.	(6, 7)

We choose $p = 3$ and form the new design as follows:

(1, 2, 3, 4, 5, 6) repeated 3 times and other blocks are

4. (1, 7)	6. (3, 7)	8. (5, 7)
5. (2, 7)	7. (4, 7)	9. (6, 7)

The parameters are

$$t^* = 7,\ b^* = 9,\ r^*_i = \begin{cases} 4 \text{ if } i = 1, \ldots, 6 \\ 6 \text{ if } i = 7 \end{cases}$$

$$K_j = \begin{cases} 6 & \text{for } j = 1, 2, 3 \\ 2 & \text{otherwise.} \end{cases}$$

The incidence matrix N will be

N =

	1	2	3	4	5	6	7	8	9
1	1	1	1	1	0	0	0	0	0
2	1	1	1	0	1	0	0	0	0
3	1	1	1	0	0	1	0	0	0
4	1	1	1	0	0	0	1	0	0
5	1	1	1	0	0	0	0	1	0
6	1	1	1	0	0	0	0	0	1
7	0	0	0	1	1	1	1	1	1

The C - matrix for the design is

C =

	1	2	3	4	5	6	7
1	3	-½	-½	-½	-½	-½	-½
2	-½	3	-½	-½	-½	-½	-½
3	-½	-½	3	-½	-½	-½	-½
4	-½	-½	-½	3	-½	-½	-½
5	-½	-½	-½	-½	3	-½	-½
6	-½	-½	-½	-½	-½	3	-½
7	-½	-½	-½	-½	-½	-½	3

Example 3.2.3

Consider example 4.1 (Non Uniform Blocks) given by Onukogu in his construction of Optimal Randomized Incomplete Block Designs (RIBD) with unequal blocks sizes.

The parameters are

$t \quad = \quad [A(3) \times B(3) \times C(2) = 8,\ b = 4,\ R_1 = 7,\ R_2 = 5,\ R_3 = K_4 = 3,\ r = 1.$

The plan was

Block

b_1	b_2	b_3	b_4
221	220	200	020
201	121	120	120
021	211	011	111
001	101		
100	110		
010			
000			

The C - matrix corresponding to the design is

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	$6/7$	$-1/7$	$-1/7$	0	0	$-1/7$	$-1/7$	0	0	0	0	0	0	$-1/7$	0	0	0	$-1/7$
2	$-1/7$	$6/7$	$-1/7$	0	0	$-1/7$	$-1/7$	0	0	0	0	0	0	$-1/7$	0	0	0	$-1/7$
3	$-1/7$	$-1/7$	$6/7$	0	0	$-1/7$	$-1/7$	0	0	0	0	0	0	$-1/7$	0	0	0	$-1/7$
4	0	0	0	$2/3$	0	0	0	0	0	0	$-1/3$	0	$-1/7$	0	0	0	0	0
5	0	0	0	0	$2/3$	0	0	0	0	$-1/3$	0	0	0	0	$-1/3$	0	0	0
6	$-1/7$	$1/7$	$-1/7$	0	0	$6/7$	$-1/7$	0	0	0	0	0	0	$-1/7$	0	0	0	$-1/7$
7	$-1/7$	$-1/7$	$-1/7$	0	0	$-1/7$	$6/7$	0	0	0	0	0	0	$-1/7$	0	0	0	$-1/7$
8	0	0	0	0	0	0	0	$4/5$	$-1/5$	0	0	$-1/5$	0	0	0	$-1/5$	$-1/5$	0
9	0	0	0	0	0	0	0	$-1/5$	$4/5$	0	0	$-1/5$	0	0	0	$-1/5$	$-1/5$	0
10	0	0	0	0	$-1/3$	0	0	0	0	$2/3$	0	0	0	0	$-1/3$	0	0	0
11	0	0	0	$-1/3$	0	0	0	0	0	0	$2/3$	0	$-1/3$	0	0	0	0	0
12	0	0	0	0	0	0	0	$-1/5$	$-1/5$	0	0	$4/5$	0	0	0	$-1/5$	$-1/5$	0
13	0	0	0	$-1/3$	0	0	0	0	0	0	$-1/3$	0	$2/3$	0	0	0	0	0
14	$-1/7$	$-1/7$	$-1/7$	0	0	$-1/7$	$-1/7$	0	0	0	0	0	0	$6/7$	0	0	0	$-1/7$
15	0	0	0	0	$-1/3$	0	0	0	0	$-1/3$	0	0	0	0	$2/3$	0	0	0
16	0	0	0	0	0	0	0	$-1/5$	$-1/5$	0	0	$-1/5$	0	0	0	$4/5$	$-1/5$	0
17	0	0	0	0	0	0	0	$-1/5$	$-1/5$	0	0	$-1/5$	0	0	0	$-1/5$	$4/5$	0
18	$-1/7$	$-1/7$	$-1/7$	0	0	$-1/7$	$-1/7$	0	0	0	0	0	0	$-1/7$	0	0	0	$6/7$

Though the C - matrix above satisfied the condition $\mathbf{C}\underline{1} = \underline{0}$ yet the other conditions of balanced design like

$C_{ii} = a \quad i = 1, 2, \ldots, 18$

$C_{ii^1} = c \quad i \neq i^1 = 1, 2, \ldots, 18$ were not satisfied hence the design is not balanced.

The balanced form with equal block sizes will be $t = 18$, $b = 153$, $r = 17$, $k = 2$ and $\lambda = 1$ but applying the method proposed above, the balanced form with unequal replications and unequal blocks sizes will have the following parameters

$$t = 18,\ b = 101,\ r_i = \begin{cases} 13 \text{ if } i = 1, 2, \ldots, 10 \\ 15 \text{ if } i = 11, \ldots, 16 \\ 17 \text{ if } i = 17, 18 \end{cases}$$

$$K_j = \begin{cases} 10 & \text{if } j = 1, 2, \ldots, 5 \\ 6 & \text{if } j = 6, 7 \ \& \ 8 \\ 2 & \text{otherwise.} \end{cases}$$

C - matrix will be

$$\mathbf{C}_{ii} = 8.5$$

$$\mathbf{C}_{ii'} = -0.5$$

3.3 EQUIREPLICATE BALANCED BLOCK DESIGNS WITH UNEQUAL BLOCK SIZES

The consideration of section 3.2 was on construction of balanced incomplete block designs with unequal block sizes and unequal replications. In this section, we shall present a method of constructing equireplicate balanced block designs.

The method formulated in the last section can be extended by choosing integers $P_1, P_2, \ldots, P_s$ with $Kp < t$ and

$$P = \sum_{\alpha=1}^{s} P_\alpha$$

Divide Kp distinct treatments into $Kp_1, Kp_2, \ldots$, kPs groups of distinct treatments. Take these $kP\propto$ treatments in a block and repeat this block $\lambda_{0\propto}P_\propto$ times where

$$\lambda_{0\propto} = \binom{kP_\propto - 2}{k-2}$$

$\propto = 1, 2, \ldots, s$. The blocks of size k are obtained from the old BIBD blocks

$$b = \binom{t}{k}$$

after deleting all the blocks

$$b_{0\propto} = \binom{kP_\propto}{k}$$

containing k treatments of $kP\propto$ chosen treatments for $\propto = 1, 2, \ldots, s$. The new design will be variance balanced design. It may be noted that if $P_\propto = P_0$ for all $\propto = 1, 2, \ldots$, s and $t = K_s$ and P_0, then one can obtain equireplicate variance balanced design.

Example 3.3.1

Using the illustrative example (example 3.2.1) presented in section 3.2, The BIB design is ($t = 8$, $b = 28$, $k = 2$, $r = 7$, $\lambda = 1$).

In this case we take $P = 2$, $s = 2$ and $k = 2$ hence by this method the new design will have 20 blocks

1.	(1, 2, 3, 4)	6.	(1, 6)	11.	(2, 7)	16.	(3, 8)
2.	(1, 2, 3, 4)	7.	(1, 7)	12.	(2, 8)	17.	(4, 5)
3.	(5, 6, 7, 8)	8.	(1, 8)	13.	(3, 5)	18.	(4, 6)
4.	(5, 6, 7, 8)	9.	(2, 5)	14.	(3, 6)	19.	(4, 7)
5.	(1, 5)	10.	2, 6)	15.	(3, 7)	20.	(4, 8)

hence the parameters are $t^* = 8$, $b^* = 20$, $r^* = 6$,

$$K^*_j = \begin{cases} 4 & \text{for } j = 1, 2, 3, 4 \\ 2 & \text{otherwise.} \end{cases}$$

The C - matrix is

$$C_{ii} = 3.5 \quad i = 1, 2, \ldots, 8$$

$$C_{ii^1} = -0.5 \quad i \neq i^1 = 1, 2, \ldots, 8$$

Restrictions on the method

(1) If the treatment t is a prime number then, an equireplicate balanced design will not exist with this construction method hence it is not possible to construct an efficiency balanced binary design with a prime number of treatments.

(2) If the block size k of the equiblock BIBD is not a factor of the treatment t, then an equireplicate balanced design will not exist hence the design will not be both efficiency balanced and variance balanced.

(3) If KP = t then the resultant design will be an equireplicate, equiblock sized complete block design.

(4) The method cannot be applied to known block sizes without altering them.

3.4 EQUIREPLICATE BALANCED BLOCK DESIGNS WITH FIXED BLOCK SIZES

The method proposed in section 3.3 produces the sizes of various blocks but if the experimenter wants to retain the existing sizes, the method then fails.

In this section, a method for constructing an equireplicate balanced block designs with known block sizes is presented. The primary objective of this method is to determine the number of blocks, number of replications and the configuration of the treatments that will achieve balancing.

Given the parameters t, b, r = 1 and k_j (j = 1, 2, ... b), we can get r^* and b^* to achieve balance by the following.

The jth block b_j of size k_j is replicated

$$\binom{t}{k_j}$$

times for all j, such that every pair of treatments appear the same number of times λj in the blocks

$$\binom{t}{k}.$$

The number of replications for the design will be

$$r^* = \sum_j \left[\binom{t}{k_j} - \binom{t-1}{k_j}\right]$$

hence the parameters for the design will be

$$t^* = t,\ b^* = \sum_j \binom{t}{k_j},$$
$$r^* = \sum_j \left[\binom{t}{k_j} - \binom{t-1}{k_j}\right],\ k_j$$

The shortcoming of this method, like most other methods of constructing BIBD, is that the number of blocks is usually large.

Example 3.4.1

Given $t = 5$, $b = 2$, $k_1 = 2$, $k_2 = 3$.

Let the treatments be 1, 2, 3, 4, 5. The plan for the above is

b_1	b_2
1	2
3	5
4	

Applying the method above, we have $b^* = 20$, $r^* = 10$.
The incidence matrix **N** is

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	1
1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0
2	1	1	1	0	0	0	1	1	1	0	1	0	0	0	1	1	1	0	0
3	1	0	0	1	1	0	1	1	0	1	0	1	0	0	1	0	0	1	1
4	0	1	0	1	0	1	1	0	1	1	0	0	1	0	0	1	0	1	0
5	0	0	1	0	1	1	0	1	1	1	0	0	0	1	0	0	1	0	1

$$K^{-1} = \left[\begin{array}{c:c} \tfrac{1}{3}I_{10} & 0I_{10} \\ \hdashline 0I_{10} & \tfrac{1}{2}I_{10} \end{array}\right]$$

C - matrix is

$$C_{ii} = 6$$

$$C_{ii'} = -\tfrac{3}{2} \quad i \neq i' = 1, 2, 3, 4, 5, 6.$$

Example 3.4.2

Let us consider example 3.2.3. Given the block design with the parameters $t = 18$, $b = 4$, $k_1 = 7$, $k_2 = 5$, $k_3 = k_4 = 3$, $r = 1$.

Applying the method proposed in this section, we have a balanced design with the parameters $t = 18$, $b = 42024$, $r = 25398$

$$C_{ii} = 23063.333$$

$$C_{ii}{}^{I} = -1356.667$$

3.5 EFFICIENCY BALANCED DESIGNS

It is known (Raghavarao (1971) theorem 4.52) that among the class of connected designs, the balanced block design is the most efficient design. Here by balance we mean variance balance. Another useful concept of balancing is the efficiency balanced design.

The efficiency of a design (a randomized block design, Onukogu (1995)) is the reciprocal of the ratio of the variance of the estimate $\hat{t}$ of a single treatment $\hat{t}_i$ or a simple contrast $\hat{t}_i - \hat{t}i^I$, obtained for the design to the variance of the same effect or contrast from a randomized complete block design (RCBD) with equal number of plots. Thus for a randomized incomplete block design, the efficiency is given by

$$\text{FRIBD / RCBD} = \begin{cases} \text{Var }(t_i/\text{RCBD})/\text{Var }(t_i/\text{RIBD}) \text{ ....} \\ \\ \text{Var }(t_i - t_i^1/\text{RCBD}/\text{Var }(t_i - t_i^1/\text{RIBD}) \end{cases} \qquad (3.4.1)$$

If the two designs do not have equal number of plots the efficiency is given by

$$\text{FRIBD / RCBD} = \left\{ \frac{(V^{RCBD} + 1)\ (V_{RIBD} + 3)\ \text{Var}(t_i/\text{RCBD})}{(V_{RCBD} + 3)\ (V_{RIBD} + 1)\ \text{Var}(t_i/\text{RIBD})} \right. \qquad \ldots(3.4.2)$$

where V_{RCBD} and V_{RIBD} are respectively the error degrees of freedom for RCBD and RIBD.

Onukogu (1995) also established the following facts on relative information

i. Given a RIBD with parameters t, b, (r_i), (k_j) (n_{ij})

$$FRIBD/RCBD = \left(\frac{n}{n-r_i}\right) \left(1 - r_i^{-1} \sum_{j(i)} n_{ij}^2/k_j\right) \text{ ....} \qquad (3.4.3)$$

$$n = \sum_{ij} n_{ij} = \sum_i r_i = \sum_j k_j$$

ii. Given a balanced incomplete block design with parameters t, b, r, k, λ; (n_{ij})

$$F_{BIBD}/_{RCBD} = \lambda t/kr$$

The proof for all these statements were provided by Onukogu (1995).

By an Efficient - Balanced design, is meant one in which all contrasts are estimated with the same efficiency. A sufficient condition is found in the off diagonal elements of the matrix P defined in equation (2.2.3) as $P = NK^{-1} N^{t}$.

For a connected binary equireplicate balanced design, P is given by Atiqullah (1961) as

$$P = \begin{bmatrix} b/t & \frac{tr-b}{t(t-1)} & \cdots\cdots & \frac{tr-b}{t(t-1)} \\ \frac{tr-b}{t(t-1)} & b/t & \cdots\cdots & \frac{tr-b}{t(t-1)} \\ \vdots & \vdots & & \vdots \\ \frac{tr-b}{t(t-1)} & \frac{tr-b}{t(t-1)} & \cdots\cdots & b/t \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \qquad (3.4.4)$$

The efficiency factor $(1 - \mu)$ relative to a randomized block with unequal block sizes is given by

$$E.F. = (1-\mu) = \frac{\sum_i r_i - \sum_j \frac{1}{k_j} \sum_i n_{ij}^2}{n(1-1/t)} \quad \ldots\ldots\ldots\ldots \qquad (3.4.5)$$

for binary design

$$E.F. = \frac{n-b}{n(1-1/t)} \quad \ldots\ldots\ldots\ldots\ldots \qquad (3.4.6)$$

and if the design is equireplicate

$$E.F. = \frac{tr-b}{r(t-1)} \quad \ldots\ldots\ldots\ldots\ldots \qquad (3.4.7)$$

Let P be a vector whose ith element is

$$\sqrt{r_i}\,, \quad and \quad let$$

$$A = P^{-1}\, C\, P^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.8)$$

where $P^{-1} = \mathrm{diag}\,({}^1/p_1,\ \ {}^1/p_2,\ \ldots,\ {}^1/p_t)$

A block design is said to be efficiency balanced if and only if [Williams (1975)]

$$A = (1 - \mu)\,(1 - {}^1/_n\, PP^1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.9)$$

that is

$$C = (1 - \mu)\,(R - {}^1/_n\, PP^1\, \underline{r}\, \underline{r}^1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.10)$$

where for binary design

$$1 - \mu = \frac{n(n-b)}{n^2 - r'r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.4.11)$$

The following lemmas on efficiency balanced block designs are very important.

Lemma 1 [Kageyama (1981)].

For an efficiency balanced block design

$M_0 = \mu\, \{I_t - ({}^1/_n)\, \underline{1}\, \underline{r}^1\}$ if and only

If $C = (1 - \mu)\, \{R - ({}^1/_n)\, \underline{r}\, \underline{r}^1\}$

Lemma 2 (Puri & Nigam (1977)}

For an equireplicate block design, the concept of 'efficiency balance' coincide

Lemma 3 {Kageyama (1981)}

An efficiency balanced binary block design with constant block size is a balanced Incomplete Block Design (BIBD).

Example 3.5.1

Let us consider the BIB in example 3.4.1. with the parameters

$t = 5, b = 20, r = 10$

$$K_j = \begin{cases} 3 \text{ for } j = 1, .., 10 \\ 2 \text{ for } j = 11, \ldots, 20 \end{cases}$$

The design is efficiency balanced with the efficiency factor of 0.75.

THEOREM 3.5.1 [(PURI AND NIGAM (1977)]

If a design is pairwise (efficiency) balanced, this does not imply that it is also efficiency (pairwise) balanced.

Proof: [cf. Puri and Nigam (1977) theorem 3].

The above theorem shows that pairwise balanced is neither necessary nor sufficient for a block design to have efficiency balance.

CHAPTER FOUR

SUMMARY AND CONCLUSIONS

4.1 SUMMARY AND CONCLUSION

In this work, we considered the block designs with t treatments, b - blocks equal and unequal replications r_i and unequal block sizes k_j. The consideration was on binary connected designs.

A block design, is the arrangement of t varieties in b blocks where the *ith* variety is replicated r_i times and the jth block contains k_j units. The reduced normal equations for estimation of the vector of variety effects $\propto$ are known to be of the form Q - C $\hat{\alpha}$ and an incomplete block design is said to be balanced if the diagonal elements of C are all equal and the off diagonal elements are also equal. From statistical point of view, variance balance is more interesting notion of balance. It follows from Kiefer (1975) that among all designs with the same grouping of experimental units, a variance balanced design is universally optimal. Hedayat and Stufken (1989) had provided the relation between pairwise balanced and variance balanced designs. A block design is said to be efficiency balanced if all the canonical efficiency factors are all equal that is $e_1 = e_2 = \ldots = e_t = 1 - \mu$ where μ is the relative loss in information. The C - matrix was given as $C = H(I - {}^1/t\ \underline{11}^1)$

where

$$H = \frac{\sum_i r_i - \sum_j \frac{1}{kj} \sum_i n_{ij}^2}{t-1}$$

and the efficiency factor (E.F.) relative to a randomized block with the same number of experimental units

$$n = \sum_i r_i = \sum_j k_j$$

is given by

$$E.F. = \frac{1}{\mu} = \frac{\sum_i r_i - \sum_i \frac{1}{k_j} \sum_i n_{ij}^2}{n(1-\frac{1}{t})}$$

4.2 RECOMMENDATIONS

The following points are recommended for future research in the balancing of incomplete block design.

1. The future researcher should include the non-binary balanced design.
2. Researcher on balanced design should find out if there is any way to reduce the number of blocks (and number of replications) and still achieve balance.
3. Since balanced design is a desirable property for its high efficiency in experimental design, government and other agencies are advised to sponsor such research due to huge amount of human and material resources needed for it.

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