Contributions to Operator Theory and Applications

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CONTRIBUTIONS TO OPERATOR THEORY AND APPLICATIONS

THESIS

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by

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APPROVAL

We the undersigned hereby certify that the work reported in this thesis by Aneke Sylvanus Jude meets the requirements for the award of the degree of **Doctor of Philosophy in Mathematics** and that the work herein has not to the best of our knowledge been previously submitted for the award of any degree, diploma or certificate in any institution.

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To the memory of my father,

Gilbert Aneke

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Abstract

This thesis consists of two parts. The first part deals with existence and approximation techniques for finding solutions of operator equations or fixed points of operators belonging to certain classes of mappings. The classes of mappings studied include the *K-positive definite operators*, the *suppressive mappings* and *accretive-type mappings*. In particular, it is proved that for a real Banach space X, the equation $Au = f$, $f \in X$, where A is a Kpd operator with the same domain as K, has a unique solution. An iteration process is constructed and shown to converge strongly to the unique solution of this equation. Furthermore, an *asymptotic* version of Kpd operators is introduced and studied and a convergence result is proved. Drawing from the ideas of Alber [1], Alber and Guerre-Delabriere [2, 3], suppressive and accretive-type mappings are studied in more general settings. In particular, it is proved that if K is a closed convex nonexpansive retract of a real uniformly smooth Banach space E, $T : K \to E$, a $d-$weakly contractive map such that a fixed point $x^* \in intK$ of T exists then a descent-like approximation sequence converges strongly to x^*. A related result deals with the approximation of a fixed point of T, when K is a subset of an arbitrary real Banach space E, and $F(T) := \{x \in E : Tx = x\} \neq \emptyset$. Moreover, *asymptotically* $d-$*weakly contractive mappings* are introduced and studied and convergence results are proved. The second part of the thesis deals with mathematical modelling of infectious diseases. Models for drug-resistant malaria parasites are presented both for single populations of humans and vectors and also for multigroup populations. Each of the models results in a system of nonlinear ordinary differential equations, which under suitable conditions leads to a locally stable equilibrium. The ecological significance of these equilibrium points emerges as a by-product. For the compartmental models, attention is devoted to the question of quantitative agreement with published field observations by the application of new nonlinear least squares techniques. A time dependent immunity model is formulated and used as a baseline study to investigate parameter behaviour. Furthermore, the multigroup models are studied in $\Re^n$. The ultimate intention is to extend to

infinite dimension, thereby providing a link between the analysis of these models and some well known and developed Hilbert/Banach space theory.

Contents

Chapter 1

GENERAL INTRODUCTION AND PRELIMINARIES

This thesis is divided into two parts. The contributions of the first part fall within the general area of operator theory while the second part is concerned with applications to mathematical modelling of communicable diseases, in particular, malaria models. For the first part, we shall, in particular, devote attention to *existence* and *approximation methods* for finding solutions of operator equations or fixed points of certain nonlinear mappings defined in a subset of a Banach space. The classes of mappings studied include: the *K−positive definite operators*, the *suppressive operators* and certain *accretive-type operators.*

It is well known that many physically significant problems can be modelled in terms of an initial value problem of the form

$$\frac{du}{dt} + Au = 0, \qquad u(0) = u_0 \tag{1.0}$$

where A is an accetive-type operator defined in an appropriate Banach space. It is clear that if the solution of equation (1.0) is independent of t, then $Au = 0$ and the solutions of this equation correspond to the *equilibrium points* of the system (1.0). Consequently, considerable research efforts have been devoted within the past half a century to finding techniques for the determination of zeros of accretive-type operators (see, for e.g., [17, 18, 19, 22, 52, 54, 58]). The study of operator equations is partly linked with fixed point theory for; u is a fixed point of the operator A if and only if u is the solution of the operator equation $(I - A)u = 0$, where I is the identity operator. The classical importance and application of fixed point theory can be seen largely in the theory of ordinary differential equations. The existence or construction of a solution to a differential equation is often reduced to the existence or location of a fixed point for an operator defined on a subset of a space of functions. In this thesis we shall employ fixed point techniques where appropriate.

Direct and iterative methods for finding solutions of operator equations or fixed points of an operator defined in an appropriate Banach space have been studied by many authors. These studies have given rise to the development of results and techniques which are now widely available in the literature.

Petryshyn [54] considered the operator equation $Au = f$, in a Hilbert space, when A is K-positive definite. Let H_1 be a dense subspace of a Hilbert space, H. An operator T with domain $D(T) \supseteq H_1$ is called *continuously* H_1 *invertible* if the range of T, $R(T)$, with T considered as an operator restricted to H_1 is dense in H and T has a bounded inverse on $R(T)$. Let H be a complex and separable Hilbert space and A be a linear unbounded operator defined on a dense domain $D(A)$ in H with the property that there exist a continuously $D(A)-$invertible closed linear operator K with $D(A) \subset D(K)$, and a constant $c > 0$ such that

$$\langle Au, Ku \rangle \geq c||Ku||^2, \qquad u \in D(A), \tag{1.0}$$

then A is called *$K-$positive definite (Kpd)* (see e.g., Petryshyn [54]). If $K = I$ (the identity operator) inequality (1.0) reduces to $\langle Au, u \rangle \geq c||u||^2$, and in this case, A is called *positive definite.* If in addition $c = 0$, A is a *positive* operator (or *accretive* operator). Positive definite operators have been studied by several authors (see e.g.[15, 17, 18, 19, 28, 38, 58]). It is clear that the class of $K-$pd operators contains among others, the class of positive definite operators, and also contains the class of invertible operators (when $K = A$) as its subclasses. Furthermore, Petryshyn

[54] remarked that for a proper choice of K, the ordinary differential operators of odd order, the weakly elliptic partial differential operators of odd order, are members of the class of K-pd operators. Moreover, if the operators are bounded, the class of Kpd operators forms a subclass of symmetrizable operators studied by Reid [58].

In [54], Petryshyn proved the following theorem.

Theorem P *If A is a Kpd operator and $D(A) = D(K)$, then there exists a constant $\alpha > 0$ such that for all $u \in D(K)$,*

$$||Au|| \leq \alpha ||Ku||.$$

Furthermore, the operator A is closed, $R(A) = H$ and the equation $Au = f$, $f \in H$, has a unique solution.

In the case that K is bounded and A is closed, F. E. Browder[19] otained a result similar to the second part of Theorem P.

In chapter two of this thesis, we extend the definition of a Kpd operator to *real Banach spaces*, X. In particular, if X is a *real separable* Banach space with a *strictly convex dual*, we prove that the equation $Au = f$, $f \in X$, where A is a Kpd operator with the same domain as K has a unique solution. Furthermore, if $X = Lp$ *(or l_p)*, $p \geq 2$, and is separable, we construct an iteration process which converges strongly to this solution.

In chapter three, we continue with the study of *Kpd* operators. We prove convergence of a simplified iteration sequence to the solution of a Kpd operator in real uniformly smooth Banach spaces. We extend the notion of a *Kpd* operator to Fréchet differentiable operators, and in this case, prove convergence of an iteration scheme to the unique solution of a *Kpd* operator equation in arbitrary Banach space. This result is also derived directly from the inverse function theorem.

Let E be a real normed linear space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^2\},$$

where $\langle .,. \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single−valued and if E^* is uniformly convex then J is uniformly continuous on bounded subsets of E (see e.g., [6]). We shall denote the single-valued duality mapping by j.

Let K be a subset of a real Banach space E. A map $T : K \to K$ is called a *strict contraction* if there exists $k \in [0, 1)$ such that $||Tx - Ty|| \leq k||x - y||$, and it is called *nonexpansive* if, for arbitrary $x, y \in K$, $||Tx - Ty|| \leq ||x - y||$. The map T is called *pseudocontractive* if, for each $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2.$$

Let K be a nonempty *convex* subset of a real normed linear space E. For strict contraction mappings, nonexpansive and pseudocontractive *self-mappings* T of K with a fixed point in K, three well known iterative methods, the celebrated *Picard method*, the *Mann iteration method* (see, for example [48]) and the *Ishikawa iteration method* (see, for example [41]) have been successfully employed to approximate such fixed points. If, however the domain of T, $D(T)$, is a *proper* subset of E (and this is the case in several applications), and T *maps* K *into* E, these iteration methods may not be well defined. Under this situation, for *Hilbert spaces*, this problem has been overcome by the introduction of the *metric projection* (see, for e.g. [20, 26]). The advantage of this is that if K is a nonempty closed convex subset of a Hilbert space H and $P_K : H \to K$ is the metric projection of H onto K, then P_K is nonexpansive. This fact which is central in most of the proofs actually characterizes Hilbert spaces and consequently, it is not available in general Banach spaces. In this connection, Alber [1] recently introduced a *generalized projection operator* Π_K in an arbitrary Banach space E which is an analogue of the metric projection in Hilbert spaces.

We recall the definition of the metric projection operator $P_K : E \to K,\ K \subset E$. The operator $P_K : E \to K$ is called a *metric projection operator* if it assigns to each $x \in E$ its nearest point $\bar{x} \in K$ i.e, $\bar{x}$ is the solution for the minimization problem

$$||x - \bar{x}|| = \inf_{\zeta \in K} ||x - \zeta||.$$

Now let E be a real normed linear space and $K \subset E$. Consider the operator $\Pi_K : E \to K$ defined by

$$\Pi_K x = \bar{x}; \qquad \bar{x} : V(x, \bar{x}) = \inf_{\zeta \in K} V(x, \zeta),$$

where

$$V(x,\zeta) := ||x||^2 - 2\langle x, j(\zeta)\rangle + ||\zeta||^2, \qquad x \in E, \zeta \in K, j(\zeta) \in J(\zeta). \tag{1.1}$$

Observe that in a Hilbert space H, (1.1) reduces to $V(x,\zeta) = ||x - \zeta||_H^2, x \in H, \zeta \in K$.

Existence and uniqueness of the operator Π_K follow from the properties of the functional $V(x,\zeta)$ and the strict monotonicity of the mapping J (see for e.g. [4]). In a Hilbert space, Π_K becomes P_K.

Using the Lyapunov functional defined in (1.1), Alber and Guerre-Delabriere [2] introduced the following classes of non-self mappings.

Let K be a nonempty subset of a Banach space E. A map $T : K \to E$ is called *strongly suppressive* on K if there exists $0 < q < 1$ such that for all $x, y \in K$,

$$V(Tx, Ty) \le qV(x, y). \tag{1.2}$$

T is called *weakly suppressive of class* $C_{\Phi(t)}$ if there exists a continuous and nondecreasing function $\Phi(t)$ defined on $\Re^+$ such that Φ is positive on $\Re^+ \backslash \{0\}$, $\Phi(0) = 0$, $\lim_{t \to +\infty} \Phi(t) = +\infty$ and $\forall x, y \in K$,

$$V(Tx, Ty) \le V(x, y) - \Phi(V(x, y)). \tag{1.3.}$$

The map T is called *nonextensive* if

$$V(Tx, Ty) \le V(x, y), \qquad \forall x, y \in K. \tag{1.4.}$$

It is easy to see that each weakly suppressive mapping is nonextensive, and each strongly suppressive operator is weakly suppresive with $\Phi(t) = (1-q)t$. In Hilbert spaces, nonextensive operators are nonexpansive and vice versa and strongly suppressive operators coincide with strict contractions.

Assuming the existence of fixed points for the above classes of operators, Alber and Guerre-Delabriere [4] proved convergence theorems with the help of generalized projection maps. They also introduced the following classes of operators.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *weakly contractive* (see e.g., [2, 3, 4]) if there exists a continuous and nondecreasing function $\Phi : [0,\infty) := \Re^+ \rightarrow \Re^+$ such that Φ is positive on $\Re^+ \setminus \{0\}$, $\Phi(0) = 0$, $\lim_{t\rightarrow\infty} \Phi(t) = \infty$ and for $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$||Tx - Ty|| \leq ||x-y|| - \Phi(||x-y||). \tag{1.5}$$

It is called *$d-$weakly contractive* if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and $\Phi(t)$ as above such that

$$|\langle Tx - Ty, j(x-y)\rangle| \leq ||x-y||^2 - \Phi(||x-y||^2). \tag{1.6}$$

The $d-$weakly contractive operators include several important classes of nonlinear operators. In particular, they include the weakly contractive operators.

In [4], Alber and Guerre-Delabriere proved, interalia, the following theorem.

Theorem AG *Let $T : G \rightarrow H$ be a $d-$weakly contractive map, G a closed convex bounded subset of a Hilbert space H and suppose that a fixed point $x^* \in int(G)$ of T exists. Then the sequence $\{x_n\}$ defined by*

$$x_1 \in G; \quad x_{n+1} := P_G(x_n - \alpha_n(x_n - Tx_n)), \quad n = 1, 2, ...,$$

where P_G is the metric projection onto the set G, $\{\alpha_n\}$ is a sequence of positive numbers such that $\sum_1^\infty \alpha_n = \infty$ and $\lim_{n\to 0}\alpha_n = 0$ converges strongly to x^*. Moreover, there exist a constant $C > 0$ and a bounded sequence $\{x_{n_l}\} \subset \{x_n\}$, l=1,2,... such that

$$||x_{n_l} - x^*||^2 \le \phi^{-1}\Big(\frac{1}{\sum_1^{n_l}\alpha_m} + C^2\alpha_{n_l}\Big),$$

Furthermore,

$$||x_{n_l+1} - x^*||^2 \le \phi^{-1}\Big(\frac{1}{\sum_1^{n_l}\alpha_m} + C^2\alpha_{n_l}\Big) + C^2\alpha_{n_l}^2,$$

$$||x_n - x^*||^2 \le ||x_{n_l+1} - x^*||^2 - \sum_{n_l+1}^{n-l}\frac{\alpha_m}{\sum_1^{m-1}\alpha_j},\ n_l + 1 \le n < n_l + 1,$$

$$||x_{n+1} - x^*||^2 \le ||x_1 - x^*||^2 - \sum_1^{n}\frac{\alpha_m}{\sum_1^{m-1}\alpha_j} \le ||x_1 - x^*||^2,\ 1 \le n \le n_1 - 1,$$

$$1 \le n_1 \le s_{max} = \max\{s : \sum_1^{s}\frac{\alpha_m}{\sum_1^{m}\alpha_j} \le ||x_1 - x^*||^2\}.$$

For a Banach space E, the *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\Big\{\frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1, ||y|| = \tau\Big\},\quad \tau > 0.$$

E is said to be uniformly smooth if $\lim_{\tau\to 0+}\frac{\rho_E(\tau)}{\tau} = 0$. Typical examples of such spaces are the *Lebesgue L_p, the sequence l_p*, and the *Sobolev W_p^m* spaces, $1 < p < \infty$.
It is known (see, e.g. [38, 39, 63]) that $\rho_E(0) = 0$. Moreover, $h_E(\tau) = \tau^{-1}\rho_E(\tau)$ is continuous, nondecreasing and $h_E(0) = 0$.

In chapter four of this thesis, we extend Theorem AG in various directions. In particular, we prove that Theorem AG remains true in *real uniformly smooth Banach spaces* and *without the boundedness condition imposed on the domain.* Furthermore, we prove a related convergent theorem in our general setting when the fixed point x^* of T exists but is not necessarily in the interior of G. Finally we prove a convergent theorem for approximating a fixed point of a uniformly continuous $d-$weakly contractive and bounded *self-map* T of G with $F(T) \neq \emptyset$, *in arbitrary real*

Banach spaces.

In 1972, Goebel and Kirk [39] introduced a class of mappings generalizing the class of nonexpansive operators. Let K be a nonempty subset of a normed space E. A mapping $T: K \to K$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$, $k_n \geq 1$, such that $\lim_{n\to\infty} k_n = 1$, and $||T^n x - T^n y|| \leq k_n ||x - y||$ for each x, y in K and for each integer $n \geq 1$.
Later in 1993, Bruck *et al* introduced and studied another class of asymptotic nonexpansive maps. A mapping $T: K \to K$ is called *asymptotically nonexpansive in the intermediate sense* (see e.g., Bruck *et al* [22]) provided T is uniformly continuous and

$$\limsup_{n\to\infty} \left\{ \sup_{x,y\in K} (||T^n x - T^n y|| - ||x - y||) \right\} \leq 0.$$

Asymptotic *pseudocontractive operators* have also been introduced and studied, first by Schu (see e.g., [59]) and then by a host of other authors, as a generalization of asymptotic nonexpansive maps. $T: K \to K$ is called *asymptotically pseudocontractive* if there exists a sequence $\{k_n\}$, $k_n \geq 1$, $\lim k_n = 1$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n ||x - y||^2$$

for each x, $y \in K$.
It is easy to see that asymptotically pseudocontractive maps include the asymptotic nonexpansive ones. These classes of maps have been studied by various authors.

Motivated by Goebel and Kirk [39], Bruck *et al* [22] and Schu [59] , we introduce and study, in chapter five, the class of *asymptotically* $d-$weakly contractive maps.
A map T with domain $D(T)$ and range $R(T)$ in E will be called *asymptotically $d-$weakly contractive* if there exists a continuous and nondecreasing function $\Phi: [0, \infty) := \Re^+ \to \Re^+$ such that Φ is positive on $\Re^+ \backslash \{0\}$, $\Phi(0) = 0$, $lim_{t\to\infty}\Phi(t) = \infty$ and for x, $y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$(*) \qquad |\langle T(PT)^{n-1}x - T(PT)^{n-1}y,\ j(x - y)\rangle| \leq k_n ||x - y||^2 - \Phi(||x - y||^2), \forall\ n \geq 1,$$

where $\{k_n\}$ is a real sequence such that $k_n \geq 1, lim_{n\to\infty}k_n = 1$ and P is a *retraction from* $R(T)$ *to* $D(T)$.

Clearly, if T is a *self-map*, then condition (*) reduces to the following one:

$$|\langle T^n x - T^n y, j(x-y)\rangle| \leq k_n||x-y||^2 - \Phi(||x-y||^2).$$

With the help of the $d-$weakly contractive maps, we prove a version of Theorem AG in real uniformly smooth Banach spaces. A corollary of our theorem extends Theorem AG from Hilbert spaces to real uniformly smooth Banach spaces. Furthermore, the *boundednes* requirement on the domain in Theorem AG is dispensed with in our more general setting. A related result deals with the approximation of fixed points of uniformly continuous asymptotically $d-$weakly contractive *self-maps* with nonempty fixed point sets, in arbitrary real Banach spaces.

Part two

Malaria is a mosquito-borne infection caused by protozoa of the genus plasmodium. Four species of the parasite, namely: P. falciparum, P. vivax, P. ovale, and P. malariae infect humans. Malaria remains the most important of the tropical diseases, being widespread throughout the tropics, but also occurring in many temperate regions.

The parasites are transmitted by the bite of infected female mosquitoes of the genus Anopheles. Mosquitoes become infected by feeding on the blood of infected people, and the parasites then undergo another phase of reproduction in the infected mosquito. Clinical symptoms such as fever, pains, and sweats may develop a few days after an infected mosquito bite.

In many parts of Africa, where malaria has long been highly endemic, people are infected so frequently that they develop a degree of acquired immunity, and may become asymptomatic carriers of the infection [36]. Treatment and control have become more difficult in recent years with the spread of drug resistant strains of malaria parasites [14, 36, 62]. Drugs such as chloroquine, nivaquine, quinine, and fansidar are used for treatment. More recent and more powerful drugs include mefloquine, and halofantrine.

It is estimated that 267 million people are presently infected, with 107 million clinical cases annually; the number of countries affected is put at 103 [62].

The biology of the four species of plasmodium is generally similar and consists of two discrete phases-sexual and asexual . The parasite migrates to the liver where it remains latent for several days while replicating. The latent period is followed by penetration of red blood cells and asexual replication within them. Asexual parasites in the blood, after surviving some developmental period, give rise to sexual stages called gametocytes. Gametocytes can remain in the blood for more than two years [36].

The emergence of drug-resistant strains of malaria parasite has become a significant health problem. Recent pronouncements by the World Health Organisation indicate the availability of strains that are resistant to virtually all known drugs.

Among the four species of plasmodium, P. falciparum causes the most serious illness and it is the most widespread in the tropics. This paper therefore focuses on the dynamics of P. falciparum malaria, although the analysis is similar for all forms of malaria infections.

An early fundamental model in the art of mathematical modelling of malaria, due to Ross-Macdonald describes the basic features of the interaction between infected humans (y) and infected mosquito vectors (q). The model is defined as follows:

$$\begin{aligned} \dot{y} &= \alpha q(1-y) - ry, \\ \dot{q} &= \beta y(1-q) - \mu q, \end{aligned}$$

where $\alpha = b\beta\frac{M}{N}$, β, r, μ are some constants defined as follows (see for e.g. [12]):

N is the size of the human population;

M is the size of the female mosquito population;

$\frac{M}{N}$ is the number of female mosquitoes per human host;

β is the rate of biting on man by a single mosquito (number of bites per unit time);

b is the proportion of infected bites on man that produce an infection;

r is the per capita rate of recovery for humans ($\frac{1}{r}$ is the average duration of infection in the human host);

μ is the per capita mortality rate for mosquitoes ($\frac{1}{\mu}$ is the average lifetime of a mosquito).

In this simple model, the total population of both humans and vectors is assumed fixed, so that the dynamical variables (y and q) are the proportion infected in each population. The first equation describes changes in the proportion of humans infected. New infections are acquired at a rate that depends on the following factors:

(i) the number of mosquito bites per person per unit time ($\beta \frac{M}{N}$)

(ii) the probability that the biting mosquito is infected (q)

(iii) the probability that a bitten human is uninfected ($1 - y$)

(iv) the probability that an uninfected person thus bitten will actually become infected (b).

Infections are lost by infected people returning to the uninfected class, at a characteristic recovery rate r. Similarly, the second equation describes changes in the proportion of mosquitoes infected. Population changes are determined by the following factors:

(i) the number of bites per mosquito per unit time (β)

(ii) the probability that the biting mosquito is uninfected ($1 - q$)

(iii) the probability that the bitten human is infected (y).

The loss term (μq) arises from the death of infected mosquitoes. The loss terms for the infected humans and infected vectors both involve death and recovery. For human hosts the recovery rate is typically faster than the death rate, whereas for vectors the opposite is the case. The origin is a local asymptotic equilibrium for this model if $\mu r > \alpha\beta$. Thus infection dies out if the product of the death rates (μ and r) for the two populations is large in the sense that $\mu r > \alpha\beta$. Thus the above formulation is a sensible approximation.

However, this model is highly simplified. The model simply assumes that an infected individual either recovers to join the susceptible group $N(1 - y)$ or dies. It fails to distinguish between the various infected categories of human and vector hosts. Thus it cannot describe accurately the recent trend in malaria infection.

This basic model has been studied, modified and generalized in different directions by various authors (see for e.g., [12, 13, 14, 47]). Aron and May [12] extended this model by introducing another population group $z-$ the latent infected humans (infected but not yet infectious). They conjectured that if the incubation interval in the mosquito has duration τ the second equation in the basic model could be replaced by the two dynamical systems:

$$\begin{aligned}\dot{z} &= ay(1-q-z) - a\hat{y}(1-\hat{q}-\hat{z})e^{-\mu\tau} - \mu z,\\ \dot{q} &= a\hat{y}(1-\hat{q}-\hat{z})e^{-\mu\tau} - \mu q,\end{aligned}$$

where the circumflex denotes evaluation at time τ in the past: $\hat{y} = y(t-\tau)$; etc. Here, the two categories of mosquito (uninfected and infected-and-infectious) are now replaced by three categories: a proportion, $1-q-z$, that are uninfected; a proportion q that are infected and infectious; and a third, new proportion z that are latent (infected but not yet infectious).

Bailey [13] considered two interacting populations–human hosts and mosquito vectors with each group consisting of three subgroups, viz susceptibles, infectives, and isolated (recovered and immune). These are designated by x, y, and z respectively for the human populations and x', y', and z' respectively for the vector populations. It follows that the number of new infections occurring in the human population in time interval δt is $\beta xy'\delta t$, where β is the infection rate. Since the converse arrangement is required to hold for vectors, namely that susceptible vectors are infected by human infectives, the number of new infections occurring in susceptible vectors in time interval δt is therefore given by $\beta' x' y \delta t$. If in addition the overall removal rates for the two populations are assumed to be γ and γ', respectively, then the numbers of removals occurring in time δt are $\gamma y \delta t$ and $\gamma' y' \delta t$ for humans and vectors, respectively. The system of differential equations for the dynamic process involved is given as

$$\begin{aligned}\dot{x} &= -\beta xy', & \dot{x}' &= -\beta' x' y + \gamma' y',\\ \dot{y} &= \beta xy' - \gamma y, & \dot{y}' &= \beta' x' y - \gamma' y',\\ \dot{z} &= \gamma y,\end{aligned}$$

where β, β', γ, γ' are some constants. Due to its relatively short life-cycle, isolation by immunity is negligible in the mosquito vectors, and hence the only isolation process is by death which is assumed to occur equally in all groups. Hence $\dot{z}' \equiv 0$ (see [14], pp. 61–68).

Let the initial states at $t = 0$ be $(x_o, y_o, 0)$ and $(x'_o, y'_o, 0)$. It is clear from the above equations that for a proper epidemic outbreak to occur, $\dot{y}_o > 0$ and $\dot{y}_o{}' > 0$. That is

$$\beta x_o y'_o > \gamma y_o \quad \text{and} \quad \beta' x'_o y_o > \gamma' y'_o.$$

Let N and N' be the populations of humans and vectors respectively, so that $N = x + y$ and $N' = x' + y'$. It follows that if both y_o and y'_o are small, then $x_o \approx N$ and $x'_o \approx N'$. Thus

$$\beta N y'_o > \gamma y_o \quad \text{and} \quad \beta' N' y_o > \gamma' y'_o.$$

Eliminating y_o we have

$$\beta N y'_o > \frac{\gamma\gamma' y'_o}{\beta' N'}.$$

Thus for an epidemic build-up to occur we must have the condition

$$NN' > \frac{\gamma\gamma'}{\beta\beta'}.$$

This model, like the first, also describes the basic interaction between the infected human host population and the mosquito vector population but with an additional group in each population.

Proposition PS [55] *Consider the system*

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= g(x_2) \end{aligned}$$

with $f(.,.)$ and $g(.)$ continuous throughout a compact subset E of $\mathcal{R}^2$. Define $P(E)$ as the projection of E onto the x_2 axis. Assume

1. *E is positively invariant for the system,*

   and

2. $x_2 = \hat{x}_2$ *is an equilibrium point that is globally asymptotically stable on a subset* A *of* $P(E)$.

Then every trajectory starting in $A' = \{(x_1, x_2) : x_2 \in A\}$ *tends asymptotically to a point of the form* $(\hat{x}_1, \hat{x}_2)$, *where* $\hat{x}_1$ *is an equilibrium of* $\dot{x}_1 = f(x_1, \hat{x}_2)$.

Consider the system

$$\dot{y}_i = \left(\sum_{j=1}^{n} w_{ij} y_j (N_i - y_i) \right) - g_i y_i, \tag{1.7}$$

where w_{ij}, N_i, g_i are constants and $w_{ij} > 0$, $N_i > 0$, $g_i > 0$, $y_i \geq 0$. Let y be the vector whose components are y_i, i=1, ... ,n; A the matrix of linear terms, and $Q(y)$ the vector of quadratic terms in (1.7). Then the vector form of the system is

$$\dot{y} = Ay - Q(y). \tag{1.8}$$

Lajmanovich and Yorke (see [44]) proved that solutions to system (1.8) are globally asymptotically stable with limiting value determined by the stability modulus $s(A)$ of the matrix A. This is defined as the maximum real part of the eigenvalues of A, i.e.,

$$s(A) = max\{Re\ \lambda : \ \lambda\ an\ eigenvalue\ of\ A\}.$$

Precisely, the following theorem was proved by Lajmanovich and Yorke.

THEOREM LY *The solutions to system (1.8) approach the origin if* $s(A) \leq 0$ *and approach a unique positive equilibrium* $\hat{y}$ *if* $s(A) > 0$, *provided* $y_i(0) > 0$ *for some* i. *Furthermore in this case* $0 < \hat{y}_i(0) < N_i$ *for each i=1, 2, ... ,n.*

Throughout this part, we shall adopt the following notations where applicable.

- N denotes the total human population;
- x denotes susceptibles (the number of people that are uninfected);
- y denotes infectives (the number of people that are severely infected);

- v denotes infected vectors (the population of vectors that can transmit the disease).

Where y and Y occur simultaneously, we designate y as the number of individuals infected with parasites that are sensitive to drugs and Y as the number of individuals that are infected with the resistant parasites with or without the sensitive parasites. Also where applicable, v and V denote respectively the population of vectors carrying sensitive strains and resistant strains, with or without the sensitive strains of the parasite.

In chapter six, models describing recent trend in malaria infection are formulated. The first model deals with single strain infection while the second incorporates resistant strains. Each of the models results in a *system of nonlinear ordinary differential equations.* A version of Proposition PS is proved in $\Re^4$. Our major theorem, Theorem 3 gives conditions for the existence of different endemic states. Since major health efforts should be geared towards the elimination of resistant parasites (and hence resistant infection) our findings and conclusions enable us to forecast the trend in resistant infection.

In chapter seven, we formulate models treating both sensitive and resistant infections in single and multigroup populations. We consider a case of a homogenous interacting population groups. Theorem LY plays a major role in the analysis of our models. Results which forecast the trend of resistant infection are established.

Finally in chapter eight we consider the more general multigroup population in $\Re^n$. Local asymptotic equilibrium are obtained in $\Re^n$. A general extension theorem is proved. The main tool in this direction is the *Poincaré-Bendixson* theorem.

Chapter 2

Existence, Uniqueness and Approximation of a Solution for a K-Positive Definite Operator Equation

2.1 Introduction

Recall the definition of a $K-$positive definite operator, which is contained in our general introduction: Let H_1 be a dense subspace of a Hilbert space, H. An operator T with domain $D(T) \supseteq H_1$ is called *continuously* H_1 *invertible* if the range of T, $R(T)$, with T considered as an operator restricted to H_1, is dense in H and T has a bounded inverse on $R(T)$. Suppose H is a complex and separable Hilbert space and A is a linear unbounded operator defined on a dense domain $D(A)$ in H with the property that there exists a continuously $D(A)$-invertible closed linear operator K with $D(A) \subseteq D(K)$, and a constant $c > 0$ such that

$$\langle Au, Ku \rangle \geq c||Ku||^2, u \in D(A) \tag{2.1}$$

Then A is called $K-$*positive definite* (Kpd) (see e.g. [54]). Recall also the following theorem which was proved in [54]:

Theorem P If A is a Kpd operator and $D(A) = D(K)$, then there exists a constant $\alpha > 0$ such that for all $u \in D(K)$,

$$||Au|| \leq \alpha ||Ku||.$$

Furthermore, the operator A is closed, $R(A) = H$ and the equation $Au = f, f \in H$, has a unique solution.

In the case that K is bounded and A is closed, F.E. Browder[19] obtained a result similar to the second part of Theorem P.

It is our purpose in this chapter to extend the notion of a Kpd operator to real separable Banach spaces, X. For such an X we shall prove that the equation $Au = f, f \in X$, where A is a Kpd operator with the same domain as K, has a unique solution. This result extends Theorem P to the general Banach spaces, X. Furthermore, if $X = Lp$ (or lp, $p \geq 2$) and is separable, we shall construct an iteration process which converges strongly to this solution.

Definition 2.1.1 Let X be a Banach space and let A be a linear unbounded operator defined on a dense domain, $D(A)$, in X. An operator A will be called $K-$*positive definite* (Kpd) if there exist a continuously $D(A)-$ invertible closed linear operator K with $D(A) \subseteq D(K)$, and a constant $c > 0$ such that for $j \in J(Ku)$,

$$\langle Au, j\rangle \geq c||Ku||^2, \qquad u \in D(A). \tag{2.2}$$

2.2 Main Results

We prove the following results:

Lemma 2.1 Let X be a Banach space with a strictly convex dual and let $A : D(A) \subseteq X \to X$ be a Kpd operator with range denoted by $R(A)$. Then A has a bounded inverse on $R(A)$.

Proof From inequality (2.2) we obtain, for $x \in D(A)$,

$$c||Ku||^2 \leq \langle Ax, j(Kx)\rangle \leq ||Ax||||j(Kx)|| = ||Ax||||Kx|| \tag{2.3}$$

K is continuously $D(A)-$invertible implies there exists a constant $m > 0$ such that for every $x \in R(A)$, $x \neq 0$, $||K^{-1}x|| \leq m||x||$ so that

$$\gamma||x|| \leq ||Kx||, \qquad \text{where} \qquad \gamma = m^{-1}. \tag{2.4}$$

Furthermore, $x \neq 0$ implies $Kx \neq 0$ and so inequality (2.3) reduces to

$$c||Kx|| \leq ||Ax||. \tag{2.5}$$

This inequality also trivially holds if $x = 0$. Thus, the inequality holds for all $x \in R(A)$. From inequalities (2.4) and (2.5) we obtain,

$$c\gamma||x|| \leq ||Ax||$$

which implies that A is invertible and

$$||A^{-1}x|| \leq \alpha||x||, \qquad \text{with} \qquad \alpha = (c\gamma)^{-1} > 0,$$

establishing that A has a bounded inverse.

Lemma 2.2 Let X be a Banach space with a strictly convex dual and let $A : D(A) \subseteq X \to X$ be a Kpd operator. Suppose for all x, $y \in D(A)$,

$$\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle. \tag{2.6}$$

Then A is closeable.

Proof We shall prove that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $D(A)$ and h is an element of X such that $x_n \to 0$ and $Ax_n \to h$ as $n \to \infty$ then $h = 0$. From $c||Kx|| \leq ||Ax||$ we observe that the convergence of $\{Ax_n\}_{n=1}^{\infty}$ implies that of $\{Kx_n\}_{n=1}^{\infty}$ and since $x_n \to 0$ and K is closed we have $Kx_n \to 0$ as $n \to \infty$. Let y be an arbitrary element in $D(A)$. Then, in view of Eq. (2.6), we have,

$$\begin{aligned} \langle h, j(Ky) \rangle &= \lim_{n\to\infty} \langle Ax_n, j(Ky) \rangle \\ &= \lim_{n\to\infty} \langle Kx_n, j(Ay) \rangle = 0, \end{aligned}$$

and so, $\langle h, j(Ky) \rangle = 0$ for each $y \in D(A)$. Since K is continuously $D(A)$-invertible, K is not identically zero. This implies $h = 0$, and so A is closeable.

Remark 2.3 If $X = H-$ *a complex Hilbert space*, the hypothesis that for all $x, y \in D(A)$,

$$\langle Ax, j(Ky) \rangle = \langle Kx, j(Ay) \rangle$$

in Lemma 2.2 can be dispensed with since it can be shown that in this case, the condition is authomatically satisfied, when j is replaced with the identity operator. If, however, $X = H-$ a *real* Hilbert space, Eq. (2.6) does not necessarily hold (with j replaced by the identity operator) and so the hypothesis that it should hold must be imposed (see e.g. [54]).

Lemma 2.4 Let X be a real Banach space and let D be a dense subset of X. Then D is of second category.

Proof It suffices to show that if $D = \cup_{i=1}^{\infty} F_i$, where , for each i, F_i is a closed set in D, then there exists some $i_0 \geq 1$ such that $F_{i0} \supseteq U$, where $U \neq \emptyset$ is an open set in D. But F_i closed in D implies $F_i = D \cap A_i$, where, for each i, A_i, is a closed set in X. So,

$$D = \cup_{i=1}^{\infty} F_i = \cup_{i=1}^{\infty} (A_i \cap D). \tag{2.7}$$

Claim $X = \cup_{i=1}^{\infty} A_i$. To verify this claim, it obviously suffices to show that $X \subseteq \cup_{i=1}^{\infty} A_i$. So, let $x \in X$. Then, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in D such that $x_n \to x$ as $n \to \infty$. But $x_n \in D$ implies $x_n \in A_{io}$ for some $i_0 \geq 1$, and since A_i is closed for each i, $x \in A_{i0}$. Hence $x \in \cup_{i=1}^{\infty} A_i$ so that $X \subseteq \cup_{i=1}^{\infty} A_i$, completing the proof of the claim. Since X is complete and hence is of second category, there exists $i_* \geq 1$ such that $A_{i*} \supseteq U \neq \emptyset$, where U is some open set in X. This implies

$$A_{i*} \cap D \subseteq U \cap D \neq \emptyset.$$

Observe that $U \cap D \neq \emptyset$ because D is dense in X and U is open in X. Hence,

$$F_{i*} = A_{i*} \cap D \supseteq U \cap D \neq \emptyset,$$

where $U \cap D$ is an open set in D. Hence D is of second category. This completes the proof of the lemma.

Theorem 2.5 Let X be a real separable Banach space with a strictly convex dual and let A be a Kpd operator with $D(A) = D(K)$. Then there exists a constant $\omega > 0$ such that for $x \in D(A)$,

$$||Ax|| \leq \omega ||Kx||.$$

Furthermore, the operator A is closed, $R(A) = X$ and the equation $Ax = h$, for each $h \in X$, has a unique solution.

Proof Introduce in $D(K)$ a duality pairing and norm defined respectively by

$$\langle x, j(y) \rangle = \langle Kx, j(Ky) \rangle$$

and

$$|x|_0 = ||Kx||. \tag{2.8}$$

Since K is closed, $R(K)$ is dense, and $K^{-1} : R(K) \to X$ is a bounded linear map, it follows from Lemma 2.3 and the bounded inverse Theorem, that $R(K) = X$. Moreover, $D(K)$ becomes a Banach space with respect to the new norm defined in (2.8). We shall denote this Banach space by X_0. Furthermore, A considered as a linear operator on X_0 to X is closed. To see this observe that since A is defined on all of X_0 ($D(A) = D(K)$), it is sufficient to prove that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in X_0 such that $|x_n - x_0|_0 \to 0$ as $n \to \infty$ and $Ax_n \to h$ as $n \to \infty$ then $h = Ax_0$. But $|x_n - x|_0 \to 0$ as $n \to \infty$ implies $||K(x_n - x)|| \to 0$ as $n \to \infty$. Since K is continuously $D(K)-$invertible, $||K(x_n - x)|| \to 0$ implies $||x_n - x|| \to 0$ as $n \to \infty$, and since A is closeable in X (Lemma 2.2) we have , $||A(x_n - x)|| \to 0$ as $n \to \infty$, i.e., $Ax_n \to Ax$ in X as $n \to \infty$. Hence $Ax = h$ and so A is closed in X_0, and being defined everywhere in X_0, it follows from the Closed

Graph Theorem that A is bounded in X_0. Hence there exists a constant $\omega > 0$ such that

$$||Ax|| \leq \omega |x|_0 = \omega ||Kx||, \tag{2.9}$$

for each $x \in X_0$. Next, to prove that $A : D(A) \subseteq X \to R(A) \subseteq X$ is closed, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $D(A)$ such that $x_n \to x$ and $Ax_n \to h$ as $n \to \infty$. Inequality (2.5) then yields that $\{Kx_n\}_{n=1}^{\infty}$ converges. But since K is closed, it follows that $x \in D(K)$ and $Kx_n \to Kx$ as $n \to \infty$. From inequality (2.9) and the fact that $D(A) = D(K)$, we have

$$||A(x_n - x)|| \leq \omega ||K(x_n - x)||,$$

and since $Kx_n \to Kx$, we obtain $Ax_n \to Ax$ as $n \to \infty$. This yields $Ax = h$, so that A is a closed operator.

To prove that $R(A) = X$ and that the equation $Ax = h$ has a unique solution, it now suffices to prove (since $D(A) = D(K) = X$) that A is one-to-one. So, assume that $x_1 \neq x_2$ are two solutions of the equation $Ax = h$. Then

$$Ax_1 = Ax_2 = h \qquad \text{or} \qquad A(x_1 - x_2) = 0.$$

Using inequality (2.5), we have,

$$c||K(x_1 - x_2)|| \leq ||A(x_1 - x_2)|| = 0,$$

so that $||K(x_1 - x_2)|| = 0$ or $x_1 - x_2 = K^{-1}(0) = 0$ so that $x_1 = x_2$. Contradiction. Thus, $R(A) = X$ and the equation $Ax = h$ has a unique solution for each $h \in X$. This completes the proof of Theorem 2.5.

In the next theorem we prove that an iteration process which has been extensively studied (see e.g. [21, 27, 52, 57]) under suitable conditions, converges strongly in Lp spaces, $p \geq 2$, to the solution of $Au = f$, where A is a Kpd operator. We shall need the following lemma:

Lemma 2.6 (W.L. Bynum, [23]) Let $X = Lp$ or $lp (p \geq 2)$. Then for all x, y in X, the following inequality holds:

$$||x + y||^2 \leq (p-1)||x||^2 + ||y||^2 + 2\langle x, j(y)\rangle. \tag{2.10}$$

Proof The proof follows immediately from the following inequality which is proved in [23]: for all x, y in X,

$$(p-1)||x+y||^2 \geq ||x||^2 + ||y||^2 + 2\langle y, j(x)\rangle.$$

Replace x by y and y by $(x-y)$ in this inequality to get,

$$||x-y||^2 \leq (p-1)||x||^2 + ||y||^2 + 2\langle -x, j(y)\rangle.$$

Now, replace x by $(-x)$ to obtain the desired result. We now prove the following theorem:

Theorem 2.7 Suppose $X = Lp$ or lp, $p \geq 2$, and X is separable and suppose $A : D(A) \subseteq X \to X$ is a Kpd operator with $D(A) = D(K) = R(K)$. Define the sequence $\{x_n\}_{n=1}^{\infty}$ iteratively by

$$\begin{aligned} x_0 &\in D(K) & \quad (2.11)\\ x_{n+1} &= x_n + t_n K^{-1} r_n, \qquad n \geq 0, & \quad (2.12)\\ t_n &= \frac{\langle Br_n, j(Kr_n)\rangle}{(p-1)||Br_n||^2}, \qquad \text{where} \qquad B = KAK^{-1} \end{aligned}$$

and

$$r_n = f - Ax_n, \qquad f \in R(K). \tag{2.13}$$

If A and K commute, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique solution of $Ax = f$ in X.

Remark Equations (2.12) and (2.13) jointly imply

$$\begin{aligned} x_{n+1} &= x_n + t_n K^{-1}(f - Ax_n)\\ &= (1-t_n)x_n + t_n[x_n + K^{-1}f - K^{-1}Ax_n]. \end{aligned}$$

Thus, Equations (2.11)–(2.13) can be written compactly as $x_0 \in D(K)$, and

$$X_{n+1} = (1-t_n)x_n + t_n[K^{-1}f + (I - K^{-1}A)x_n]$$

where t_n is as defined above.

Proof of Theorem 2.7 The existence of a unique solution to $Ax = f$ follows from Theorem 2.5. Using Equations (2.12) and (2.13) and the linearity of A and K we obtain:

$$\begin{aligned} r_{n+1} &= f - Ax_{n+1} \\ &= f - A(x_n + t_n K^{-1} r_n) = (f - Ax_n) - t_n A K^{-1} r_n \\ &= r_n - t_n A K^{-1} r_n \end{aligned}$$

so that,

$$K r_{n+1} = K r_n - t_n K A K^{-1} r_n = K r_n - t_n B r_n.$$

Using inequality (2.10), we have,

$$\begin{aligned} ||K r_{n+1}||^2 &= ||K r_n - t_n B r_n||^2 \\ &\leq ||K r_n||^2 - 2t_n \langle B r_n, j(K r_n)\rangle + (p-1) t_n^2 ||B r_n||^2 \\ &= ||K r_n||^2 - 2\frac{(\langle B r_n, j(K r_n)\rangle)^2}{(p-1)||B r_n||^2} + \frac{(\langle B r_n, j(K r_n)\rangle)^2}{(p-1)||B r_n||^2} \end{aligned}$$

i.e.,

$$||K r_{n+1}||^2 \leq ||K r_n||^2 - \frac{(\langle B r_n, j(K r_n)\rangle)^2}{(p-1)||B r_n||^2}. \tag{2.14}$$

Hence, the sequence $\{||K r_n||\}_{n=1}^{\infty}$ is monotonically decreasing and so converges to some real number $\delta \geq 0$. Inequality (2.14) now implies

$$\lim_{n\to\infty} \frac{(\langle B r_n, j(K r_n)\rangle)^2}{(p-1)||B r_n||^2} = 0. \tag{2.15}$$

Since A and K commute, then $B = KAK^{-1} = AKK^{-1} = A$, and so (2.15) yields

$$\lim_{n\to\infty} \frac{(\langle A r_n, j(K r_n)\rangle)^2}{||A r_n||^2} = 0. \tag{2.16}$$

Since K is continuously $D(A)-$invertible, there exists $\beta \geq 0$ such that for each $x \in D(K)$, $\beta||x|| \leq ||Kx||$. Thus, from the definition of the Kpd operator A, we obtain

$$\begin{aligned} \frac{(\langle A r_n, j(K r_n)\rangle)^2}{||A r_n||^2} &\geq \frac{c^2 ||K r_n||^4}{||A r_n||^2} \\ &\geq \frac{c^2 \beta^4 ||r_n||^4}{||A||^2 ||r_n||^2} = \frac{c^2 \beta^2 ||r_n||^2}{||A||^2} \geq 0. \end{aligned} \tag{2.17}$$

Equation (2.16) and inequality (2.17) now imply that $r_n \to 0$ as $n \to \infty$, i.e., $Ax_n \to f$ as $n \to \infty$. Since A has a bounded inverse, this implies $x_n \to A^{-1}f$, the unique solution of $Au = f$ in X.

Remarks

1. If $K = A$, then obviously the commutativity requirement in Theorem 2.7 can be dispensed with since, in this case, $B = A$.

2. If $K = I$, the identity operator, then A and K obviously commute, and the commutativity requirement will again not be needed.

3. It is of interest to obtain a convergence result for Lp (or lp) similar to Theorem 2.7 where $1 < p < 2$.

Chapter 3

A Local Approximation Methods for the Solution of $K-$Positive Definite Operator Equations

3.1 Introduction

In chapter 2 , we extended the notion of a $K-$positive definite (Kpd) operator to *real separable Banach spaces*, X. In particular, if X is a real separable Banach space with a *strictly convex dual*, we proved that the equation $Au = f, f \in X$, where A is a Kpd operator with the same domain as K has a unique solution. Furthermore, if $X = Lp$ $(or\ lp)$, $p \geq 2$, and is separable, we constructed an iteration process which converges strongly to this solution.

C. E. Chidume and M. O. Osilike [33], extended the above result to a larger space, the *q-uniformly Banach spaces.*

Definition 3.1.1 *Let X be a Banach space and let A be a linear unbounded operator defined on a dense domain, $D(A)$, in X. The operator A will be called asymptotically $K-$positive definite Kpd*

if there exist a continuously $D(A)-$invertible closed linear operator K with $D(K) \supseteq D(A) \supseteq R(A)$, and a constant $c > 0$ such that for $j(Ku) \in J(Ku)$,

$$\langle K^{n-1}Au, j(K^n u)\rangle \geq ck_n||K^n u||^2, \qquad u \in D(A), \tag{3.1}$$

where $\{k_n\}$ is a real sequence such that $k_n \geq 1, lim_{n\to\infty}k_n = 1$.

Lemma 3.1.2(see, e.g., [56]) *Let E be a real uniformly smooth Banach space and let J be the normalized duality map on E. Then for any given $x, y \in E$, the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x)\rangle + \max\{||x||, 1\}||y||b(||y||), \ \forall j(x) \in J(x).$$

where b is a continuous nondecreasing function satisfying the conditions: $b(0) = 0$, $b(ct) \leq cb(t)$, $\forall c \geq 1$.

It is our purpose in this chapter to extend the notion of a *kpd* operator to Fréchet differentiable operators. Under this setting, a local existence theorem and an iterative scheme which converges to the unique solution of the *Kpd* operator equation in an *arbitrary Banach spaces*, are derived from the inverse function theorem. Moreover, we introduce and study a new notion-*asymptotically K-positive definite operators* (see Definition 3.1.1).

3.2 Main Results

Now, we state the inverse function theorem and sketch its proof. We derive from the proof of the theorem that the iteration scheme in Theorem 2 of [29] converges to the unique solution of $Ax = f$ in an arbitrary real Banach space, provided $||f - Ax_0||$ is sufficiently small.

Theorem 3.1 (The inverse function Theorem) *Suppose X, Y are Banach spaces and $A : X \to Y$ is such that A has uniformly continuous Fréchet derivatives in a neighborhood of some point x_0 of X. Then if $A'(x_0)$ is a linear homeomorphism of X onto Y, then A is a local homeomorphism of a neighborhood $U(x_0)$ of x_0 to a neighborhood of $A(x_0)$.*

Proof Let $A(x_0) = y_0$. We first determine ρ so that $A(x_0+\rho) = y$ provided $||y-y_0||$ is sufficiently small, or equivalently

$$A(x_0+\rho) - A(x_0) = y - y_0. \tag{3.2}$$

Since A is C^1 at x_0 and $A'(x_0)$ is invertible, then (3.2) and Taylor's Theorem imply that $A'(x_0)\rho + R(x_0,\rho) = y - y_0$, i.e.,

$$\rho = [A'(x_0)]^{-1}[(y-y_0) - R(x_0,\rho)],$$

where the remainder

$$R(x_0,\rho) = A(x_0+\rho) - A(x_0) - A'(x_0)\rho = o(||\rho||).$$

We show that (3.2) has one and only one solution for $||\rho||$ sufficiently small, by proving that the operator

$$T\rho = [A'(x_0)]^{-1}\{y - y_0 - R(x_0,\rho)\}$$

is a contraction mapping of a sphere $S(0,\epsilon)$ in X into itself, for some ϵ sufficiently small. For any $\rho_1, \rho_2 \in S(0,\epsilon)$,

$$\begin{aligned} A'(x_0)(T\rho_2 - T\rho_1) &= R(x_0,\rho_1) - R(x_0,\rho_2) \\ &= A(x_0+\rho_1) - A(x_0+\rho_2) - A'(x_0)(\rho_1-\rho_2) \\ &= \int_0^1 \{A'(x_0 + t\rho_1 + (1-t)\rho_2) - A'(x_0)\}(\rho_1-\rho_2)dt. \end{aligned}$$

Hence

$$||T\rho_2 - T\rho_1|| \leq \int ||[A'(x_0)]^{-1}||||A'(x_0+t\rho_1+(1-t)\rho_2) - A'(x_0)||||\rho_1-\rho_2||dt. \tag{3.3}$$

Since A is a C^1 mapping, the middle term of the last integral can be made arbitrarily small by choosing $||\rho_1||$, $||\rho_2||$ sufficiently small; and hence for some constant $0 \leq \alpha < 1$ (and independent of $y - y_0$) and sufficiently small $\epsilon > 0$, $||T\rho_2 - T\rho_1|| \leq \alpha||\rho_2 - \rho_1||$ for all $\rho_1, \rho_2 \in S(0,\epsilon)$. Furthermore, T maps $S(0,\epsilon)$ into itself. For, $||T\rho|| = ||T\rho - T(0)|| + ||T(0)|| \leq \alpha||\rho|| + ||T(0)||$ and $||T(0)|| = ||[A'(x_0)]^{-1}(y-y_0)|| < (1-\alpha)\epsilon$ provided $||y-y_0|| < (1-\alpha)\epsilon||[A'(x_0)]^{-1}||^{-1}$. Hence T is a contraction map of $S(0,\epsilon)$ into itself. By the contraction mapping theorem, T has a unique fixed point ρ^* in $S(0,\delta)$ where $\delta \leq \epsilon$ is chosen so small that $A(S(0,\delta) \subset S(y_0, (1-\alpha)\epsilon||[A'(x_0)]^{-1}||^{-1})$. Reversing the steps in the argument, one finds that $A(x_0+\rho) = y$ has one and only one solution when $||y-y_0||$ and $||\rho||$ are sufficiently small. Also, $A^{-1}(y) = x$ is a well-defined and continuous mapping from a sphere $S(y_0,\eta)$ in Y to X.

Corollary 3.2 *Under the conditions of Theorem 3.1, the iteration sequence*

$$x_{n+1} = x_n + [A'(x_0)]^{-1} r_n, \quad r_n = [y - A(x_n)],$$

converges to the unique solution of $A(x) = y$ in $U(x_0)$.

Proof Since the operator T in the proof of Theorem 3.1 is a contraction map, the sequence $\rho_n = T\rho_{n-1}$ converges to the unique fixed point of T. From Theorem 3.1, for $||y - A(x_0)||$ sufficiently small, $A(x) = y$ has a unique solution $x = x_0 + \rho^*$, where ρ^* is the limit of the sequence $\rho_0 = 0$, $\rho_{n+1} = T\rho_n$. It then follows that the sequence $x_n = x_0 + \rho_n$ converges to $x_0 + \rho^*$, the unique solution of $A(x) = y$ in $U(x_0)$. Now,

$$\begin{aligned}
x_n = x_0 + \rho_n &= x_0 + T\rho_{n-1} \\
&= x_0 + [A'(x_0)]^{-1}[y - A(x_0)] - R(x_0, \rho_{n-1})] \\
&= x_0 + [A'(x_0)]^{-1}[y + A'(x_0)\rho_{n-1} - A(x_0 + \rho_{n-1})] \\
&= x_0 + \rho_{n-1} + [A'(x_0)]^{-1}[y - A(x_{n-1})] \\
&= x_{n-1} + [A'(x_0)]^{-1}[y - A(x_{n-1}].
\end{aligned}$$

Henceforth, an operator A defined on a dense domain $D(A)$ of a real Banach space will be called $K-$positive definite if A is Fréchet differentiable and there exist a continuously $D(A)-$invertible closed linear operator K with $D(A) \subseteq D(K)$, and a constant $c > 0$ such that for $j \in J(Ku)$, we have

$$\langle Au, j \rangle \geq ||Ku||^2, \qquad u \in D(A).$$

Corollary 3.3 *Suppose A is a Kpd operator defined on a dense domain $D(A)$ of a real Banach space, X with range $R(T)$ in X. If for some $x_0 \in X$, $A'(x_0)$ is a linear homeomorphism of*

X onto Y, then A is a linear homeomorphism of a neighborhood $U(x_0)$ of x_0 to a neighborhood of $A(x_0)$. Furthermore, if $||y - A(x_0)||$ is sufficiently small, the sequence $x_{n+1} = x_n + K^{-1}r_n$, where $r_n = [y - A(x_n)]$ converges to the unique solution of $A(x) = y$ in $U(x_0)$.

Proof $A'(x_0)$ satisfies the condition for K in the definition of a Kpd operator. Hence setting $K = A'(x_0)$ in Theorem 3.1, we are done.

Remark 3.4 *If X is a separable Banach space, with a strictly convex dual and the operator A is linear, a global existence result was obtained in the domain of A, $D(A)$ in Theorem 1 of [29].*

Remark 3.5 *The iteration scheme $\{x_n\}$ in Corollary 3.3 above corresponds to the one of Theorem 2 in [29] by setting $t_n \equiv 1$. In Theorem 2 of [29], the scheme $x_{n+1} = x_n + t_nK^{-1}$ converges globally to the unique solution of $A(x) = y$ in Lp (or lp), $p \geq 2$, while in Corollary 3.2 above the corresponding scheme converges locally to the unique solution of $A(x) = y$ in some neighborhood of a point x_0 in a real Banach space X. Furthermore, under this setting, the operator A need not be linear but Fréchet differentiable.*

By writing our iteration scheme in the form of Theorem CO [33], we prove the following Theorem for asymptotically $K-$positive definite operators in a *uniformly smooth* Banach space.

Theorem 3.6 *Suppose X is a real uniformly smooth Banach space. Suppose A is an asymptotically $K-$positive definite operator defined in a neighborhood $U(x_0)$ of a real uniformly smooth Banach space, X. Define the sequence $\{x_n\}$ by $x_0 \in U(x_0)$, $x_{n+1} = x_n + r_n$, $n \geq 0$, $r_n = K^{-1}y - K^{-1}A(x_n)$, $y \in R(A)$. Then $\{x_n\}$ converges strongly to the unique solution of $A(x) = y \in U(x_0)$.*

Proof By the linearity of K we obtain $Kr_{n+1} = Kr_n - Ax_n$. Using Lemma 3.1.2 and Definition 3.1.1, we obtain the following estimates:

$$||K^n r_{n+1}||^2 \quad \leq \quad ||K^n r_n - K^{n-1}Ar_n||^2$$

$$\leq \quad ||K^n r_n||^2 - 2\langle K^{n-1}Ar_n, j(K^n r_n)\rangle + \max\{||K^n r_n||, 1\}||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||)$$
$$\leq \quad ||K^n r_n||^2 - 2ck_n||K^n r_n||^2 + \max\{||K^n r_n||, 1\}||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||) \qquad (3.4)$$

Since A is Fréchet differentiable and by the properties of the function b, the quantity $||A^n r_n||b(||A^{\prime}$ can be made as small as possible in a small neighborhood $U(x_0)$ of X. Infact there exists c such that

$$||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||) \leq ck_n||K^n r_n||^2. \qquad (3.5)$$

$$||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||) \leq ck_n||K^n r_n||. \qquad (3.6)$$

Inequalities (3.5) and (3.6) imply that the sequence $||K^n r_n||_{n=0}^{\infty}$ is monotone decreasing and hence converges to some real number $\beta \geq 0$. Inequalities (3.4), (3.5) and (3.6) imply that

$$\lim_{n\to\infty} ||K^n r_n|| = 0.$$

Since K is continuously $D(A)-$invertible, this implies that $r_n \to 0$. Since A has a bounded inverse, this implies $x_n \to A^{-1}y$, the unique solution of $Ax = y$ in $U(x_0)$.

Chapter 4

Approximations of Fixed points of weakly contractive Non–self Maps in Banach Spaces

4.1 Introduction

Let $K \subseteq E$ be closed convex and let P be a mapping of E onto K. Then P is said to be *sunny* if $P(Px+t(x-Px)) = Px$ for all $x \in E$ and $t \geq 0$. A mapping P of E to E is said to be a *retraction* if $P^2 = P$. If a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, *range of* P. A subset K of E is said to be a *sunny nonexpnsive retract* of E if there exists a sunny nonexpansive retraction of E onto K. *If $E = H$, the metric projection P_K is a sunny nonexpansive retraction from H to any closed convex subset of H.*

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *d-weakly contractive* if there exists a continuous and nondecreasing function $\Phi : [0, \infty) := \Re^+ \to \Re^+$ such that Φ is positive on $\Re^+ \setminus \{0\}$, $\Phi(0) = 0$, $\lim_{t\to\infty} \Phi(t) = \infty$ and for $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$|\langle Tx - Ty, j(x-y)\rangle| \leq ||x-y||^2 - \Phi(||x-y||^2). \tag{4.1}$$

It is called *weakly contractive* (see e.g., [3, 4, 5]) if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and $\Phi(t)$ as above such that

$$||Tx - Ty|| \leq ||x - y|| - \Phi(||x - y||). \tag{4.2}$$

If $F(T) \neq \emptyset$ and inequalities (4.1) and (4.2) hold for $x \in D(T)$ and $x^* \in F(T)$, then the operators will be called *$d-$weakly hemi-contractive* and *weakly hemi-contractive*, respectively. Note that, if we set $\Phi(t^2) = \psi(t)$, then ψ is a continuous and nondecreasing function from $\Re^+$ to $\Re^+$ such that ψ is positive on $\Re^+ \setminus \{0\}$, $\psi(0) = 0$, $\lim_{t\to\infty} \psi(t) = \infty$. Thus, the above definition of $d-$weakly contractive map can be restated as follows: for all $x, y \in D(T)$, there exist $j(x - y) \in J(x - y)$ and $\psi(t)$ as above such that

$$|\langle Tx - Ty, j(x - y)\rangle| \leq ||x - y||^2 - \psi(||x - y||). \tag{4.3}$$

The $d-$weakly contractive operators were first introduced and studied by Alber and Guerre-Delabriere [4] and include several important classes of nonlinear operators. In particular, they include the weakly contractive operators.

Recall the following theorem [4] (see our General Introduction).

Theorem AG *Let $T : G \to H$ be a $d-$weakly contractive map, G a closed convex bounded subset of a Hilbert space H and suppose that a fixed point $x^* \in int(G)$ of T exists. Then the sequence $\{x_n\}$ defined by*

$$x_1 \in G; \quad x_{n+1} := P_G(x_n - \alpha_n(x_n - Tx_n)), \quad n = 1, 2, ..., \, s \tag{4.4}$$

where P_G is the metric projection onto the set G, $\{\alpha_n\}$ is a sequence of positive numbers such that $\sum_1^\infty \alpha_n = \infty$ and $\lim_{n\to 0} \alpha_n = 0$ converges strongly to x^*. Moreover, there exist a constant $K > 0$ and a bounded sequence $\{x_{n_l}\} \subset \{x_n\}$, l=1,2,... such that

$$||x_{n_l} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + K^2\alpha_{n_l}\Big), \tag{4.5}$$

Furthermore,

$$||x_{n_l+1} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + K^2\alpha_{n_l}\Big) + K^2\alpha_{n_l}^2, \tag{4.6}$$

$$||x_n - x^*||^2 \leq ||x_{n_l+1} - x^*||^2 - \sum_{n_l+1}^{n-l} \frac{\alpha_m}{\sum_1^{m-1} \alpha_j}, \; n_l + 1 \leq n < n_l + 1, \tag{4.7}$$

$$||x_{n+1} - x^*||^2 \leq ||x_1 - x^*||^2 - \sum_1^{n} \frac{\alpha_m}{\sum_1^{m-1} \alpha_j} \leq ||x_1 - x^*||^2, \; 1 \leq n \leq n_1 - 1, \tag{4.8}$$

$$1 \leq n_1 \leq s_{max} = \max\{s : \sum_1^{s} \frac{\alpha_m}{\sum_1^{m} \alpha_j} \leq ||x_1 - x^*||^2\}. \tag{4.9}$$

From theorem AG, two questions arise quite naturally.

Question 1. Can the boundedness condition on G in theorem AG be dropped ?

Question 2. Can theorem AG be extended to Banach spaces more general than Hilbert spaces ?

It is our purpose in this chapter to give affirmative answers to these questions. In particular, we prove that Theorem AG remains true in *real uniformly smooth Banach spaces* and *without the boundedness condition imposed on G.* Furthermore, we prove a related convergent theorem in our more general setting when the fixed point x^* of T exists but is not necessarily in the interior of G. Finally, we prove a convergence theorem for approximating a fixed point of a uniformly continuous $d-$weakly contractive and bounded *self map* T of G with $F(T) \neq \emptyset$, *in arbitrary Banach spaces.*

4.2 Preliminaries

In the sequel we shall use the following well known Lemmas.

Lemma 4.2.1 (see e.g., [34, 65]). Let E be a real Banach space and J the normalized duality map on E. Then for any given $x, y \in E$, the following inequality holds:

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \forall j(x+y) \in J(x+y).$$

Lemma AG ([3]) *Let $\{\lambda_k\}$ and $\{\gamma_k\}$ be sequences of nonnegative numbers and $\{\alpha_k\}$ be a sequence of positive numbers satisfying the conditions $\sum_1^\infty \alpha_n = \infty$ and $\frac{\gamma_n}{\alpha_n} \to 0$, as $n \to \infty$. Let the recursive inequality*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \phi(\lambda_n) + \gamma_n, n = 1, 2, ..., \tag{4.10}$$

be given where $\phi(\lambda)$ is a continuous and nondecreasing function from $\Re^+$ to $\Re^+$ such that it is positive on $\Re^+ \setminus \{0\}$, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$. Then

(a) $\lambda_n \to 0$, as $n \to \infty$;

(b) there exists a subsequence $\{\lambda_{n_k}\} \subset \{\lambda_n\}$, l=1,2,..., such that

$$\lambda_{n_l} \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + \frac{\gamma_{n_l}}{\alpha_{n_l}}\Big), \tag{4.11}$$

$$\lambda_{n_l+1} \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + \frac{\gamma_{n_l}}{\alpha_{n_l}}\Big) + \gamma_{n_l}, \tag{4.12}$$

$$\lambda_n \leq \lambda_{n_l+1} - \sum_{n_l+1}^{n-l} \frac{\alpha_m}{A_m}, \quad n_l + 1 \leq n < \quad n_l + 1, \quad A_m = \sum_1^{m-1} \alpha_i, \tag{4.13}$$

$$\lambda_{n+1} \leq \lambda_1 - \sum_1^{n} \frac{\alpha_m}{A_m} \leq \lambda_1, \quad 1 \leq n \leq n_1 - 1, \tag{4.14}$$

$$1 \leq n_1 \leq s_{max} = \max\{s : \sum_1^{s} \frac{\alpha_m}{A_m} \leq \lambda_1\}. \tag{4.15}$$

We shall also need the following Lemma whose proof is identical with the proof of Lemma 5.6 of [4]. However, for completeness, we give a sketch of the proof.

Lemma 4.2.2 Let E be an arbitrary real Banach space and let $T : D(T) \subseteq E \to E$ be a $d-$weakly contractive map, and suppose that a fixed point $x^* \in int(K)$ of T exists. Then $A := I - T$ is bounded.

Proof Clearly A is accretive. Then by Lemma 5.5 of [4](see also [53]) there exists a constant $r_0 > 0$ and a closed ball $S(r_0, x^*) \subset D(A)$ such that *for all* $x \in D(A)$ we have

$$\langle Ax - Ax^*, j(x - x^*)\rangle \geq r_0||Ax|| - c_0(||x - x^*|| + r_0), \tag{4.16}$$

where $c_0 = \sup_{\eta \in S(r_0, x^*)} ||A(\eta)|| < \infty$. On the other hand, for some $j(x - x^*) \in J(x - x^*)$ we have that

$$\begin{aligned} \langle Ax - Ax^*, j(x - x^*)\rangle &= \langle x - x^*, j(x - x^*)\rangle - \langle Tx - Tx^*, j(x - x^*)\rangle \\ &\leq ||x - x^*||^2 + |\langle Tx - Tx^*, j(x - x^*)\rangle| \leq 2||x - x^*||^2. \end{aligned} \tag{4.17}$$

Thus from (4.16) and (4.17) we get that

$$||Ax|| \leq r_0^{-1}\Big(2||x - x^*||^2 + c_0(||x - x^*|| + r_0)\Big). \tag{4.18}$$

Hence the conclusion holds.

4.3 Main Results

Now, we state and prove the following Theorems.

Theorem 4.1 Let E be a real uniformly smooth Banach space. Suppose K is a closed convex subset of E which is a sunny nonexpansive retract of E with P as the sunny nonexpansive retraction. Suppose $T : K \to E$ is a $d-$weakly contractive map such that a fixed point $x^* \in int(K)$ of T exists. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} := P\Big(x_n - \alpha_n(x_n - Tx_n)\Big),\ n \geq 1, \tag{4.19}$$

where $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n \leq d_0$, $\{x_n\}$ converges strongly to $x^* \in F(T)$. Moreover, there exist a constant $d > 0$ and a subsequence $\{x_{n_l}\} \subseteq \{x_n\}$ such that

$$||x_{n_l} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + d\overline{\gamma}_{n_l}\Big), \tag{4.20}$$

where $\overline{\gamma}_n := ||j(p_n - x^*) - j(x_n - x^*)||$ and $p_n := x_n - \alpha_n A x_n$.

Furthermore,

$$||x_{n_l+1} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + d\overline{\gamma}_{n_l}\Big) + d\alpha_{n_l}\overline{\gamma}_{n_l}, \tag{4.21}$$

$$||x_n - x^*||^2 \leq ||x_{n+1} - x^*||^2 - \sum_{n_l+1}^{n-l} \frac{\alpha_m}{\sum_1^{m-1} \alpha_j}, n_l + 1 \leq n < n_l + 1, \tag{4.22}$$

$$||x_{n_l+1} - x^*||^2 \leq ||x_1 - x^*||^2 - \sum_1^{n} \frac{\alpha_m}{\sum_1^{m-1} \alpha_j} \leq ||x_1 - x^*||^2, 1 \leq n \leq n_1 - 1, \tag{4.23}$$

$$1 \leq n_1 \leq s_{max} = \max\{s : \sum_1^{s} \frac{\alpha_m}{\sum_1^{m} \alpha_j} \leq ||x_1 - x^*||^2\}. \tag{4.24}$$

Proof Observe that the recursion formula (3.1) can be written as follows:

$$x_{n+1} = P(x_n - \alpha_n A x_n), \quad n \geq 0, \quad \text{where} \quad A := (I - T). \tag{4.25}$$

Moreover, we have that $\langle Ax - Ax^*, j(x - x^*)\rangle \geq \Phi(||x - y||^2)$, where $\Phi(t)$ is as in (1.1). Now, choose r sufficiently large such that $x_1 \in B_r(x^*)$. Let $G := B_r(x^*) \cap K$, then since by Lemma 4.2.2 A is bounded we have that $A(G)$ is bounded. Let $diam.A(G) = \sigma$. As j is uniformly continuous on bounded subsets of E, for $\epsilon = \frac{\Phi((\frac{r}{2})^2)}{2\sigma}$ there exists a $\delta > 0$ such that $x, y \in D(T)$, $||x - y|| < \delta$ implies $||j(x) - j(y)|| < \epsilon$. Set $d_0 = min\{1, \frac{\delta}{2\sigma}, \frac{r}{2\sigma}\}$.

Claim $\{x_n\}$ is bounded. Suffices to show that x_n is in G for all $n \geq 1$. The proof is by induction. By our assumption $x_1 \in G$. Suppose $x_n \in G$. We prove that $x_{n+1} \in G$. Assume for contradiction that $x_{n+1} \notin G$. Then, since $x_{n+1} \in K$ $\forall n \geq 1$, we have that $||x_{n+1} - x^*|| > r$. Thus we have the following estimates:

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||P(x_n - \alpha_n A x_n) - Px^*|| \\ &\leq ||x_n - x^* - \alpha_n(Ax_n - Ax^*)|| \end{aligned}$$

and hence

$$\begin{aligned} ||x_n - x^*|| &\geq ||x_{n+1} - x^*|| - \alpha_n||Ax_n - Ax^*|| \\ &> r - \alpha_n\sigma \geq r - \frac{r}{2} = \frac{r}{2}. \end{aligned}$$

Set $p_n := x_n - \alpha_n A x_n$. Then from (4.19), Lemma 4.2.1 and the above estimates we have that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||P(x_n - \alpha_n(Ax_n - Ax^*)) - Px^*||^2 \\
&\leq ||x_n - x^* - \alpha_n(Ax_n - Ax^*)||^2 \\
&\leq ||x_n - x^*||^2 - 2\alpha_n\langle Ax_n - Ax^*, j(p_n - x^*)\rangle \\
&= ||x_n - x^*||^2 - 2\alpha_n\langle Ax_n - Ax^*, j(x_n - x^*)\rangle \\
&\quad -2\alpha_n\langle Ax_n - Ax^*, j(p_n - x^*) - j(x_n - x^*)\rangle \\
&\leq ||x_n - x^*||^2 - 2\alpha_n\Phi(||x_n - x^*||^2) \\
&\quad +2\alpha_n||Ax_n||||j(p_n - x^*) - j(x_n - x^*)|| \qquad (4.26)
\end{aligned}$$

Since $||p_n - x_n|| \leq \alpha_n||Ax_n|| \leq \alpha_n\sigma < \delta$ we have that $||j(p_n - x^*) - j(x_n - x^*)|| \leq \frac{\Phi((\frac{r}{2})^2)}{2\sigma}$. Thus (4.26) gives that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq ||x_n - x^*||^2 - 2\alpha_n\Phi((\frac{r}{2})^2) + 2\alpha_n\sigma\frac{\Phi((\frac{r}{2})^2)}{2\sigma} \\
&= ||x_n - x^*||^2 - 2\alpha_n\Phi((\frac{r}{2})^2) + \alpha_n\Phi((\frac{r}{2})^2) \\
&= ||x_n - x^*||^2 - \alpha_n\Phi((\frac{r}{2})^2) < r^2. \qquad (4.27)
\end{aligned}$$

i.e. $||x_{n+1} - x^*|| < r$, a contradiction. Therefore $x_{n+1} \in G$. Thus by induction $\{x_n\}$ is bounded. Now we show that $x_n \to x^*$. Note that $p_n - x_n \to 0$ as $n \to \infty$ and hence by the uniform continuity of j on bounded subsets of E we have that

$$\overline{\gamma}_n := ||j(p_n - x^*) - j(x_n - x^*)|| \to 0 \text{ as } n \to \infty. \qquad (4.28)$$

Let $\lambda_n := ||x_n - x^*||^2$ and $\gamma_n := 2\alpha_n\sigma\overline{\gamma}_n$, then from inequality (4.26) we obtain that

$$\lambda_{n+1} \leq \lambda_n - 2\alpha_n\Phi(\lambda_n) + \gamma_n, \qquad (4.29)$$

where $\frac{\gamma_n}{\alpha_n} \to 0$ as $n \to \infty$. Thus, the conclusions of the theorem follow from Lemma AG, completing the proof of the theorem.

If $x^* \in F(T)$ is an arbitrary point of $D(T)$ then we have the following theorem.

Theorem 4.2 Let K be a closed convex subset of a real uniformly smooth Banach space. Suppose K is a sunny nonexpansive retract of E with P as the sunny nonexpansive retraction. Let $T : K \to E$ be a $d-$weakly contractive bounded map with $F(T) := \{x \in G : Tx = x\} \neq \emptyset$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} := P\Big(x_n - \alpha_n(x_n - Tx_n)\Big),\ n \geq 1. \tag{4.30}$$

where $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n \leq d_0$, then, $\{x_n\}$ converges strongly to $x^* \in F(T)$. Moreover, there exist a constant $d > 0$ and a subsequence $\{x_{n_l}\} \subseteq \{x_n\}$ such that

$$||x_{n_l} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + d\overline{\gamma}_{n_l}\Big). \tag{4.31}$$

where $\overline{\gamma}_n$ is as defined in (4.28).

Furthermore,

$$||x_{n_l+1} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + d\overline{\gamma}_{n_l}\Big) + d\alpha_{n_l}\overline{\gamma}_{n_l}, \tag{4.32}$$

$$||x_n - x^*||^2 \leq ||x_{n+1} - x^*||^2 - \sum_{n_l+1}^{n-l} \frac{\alpha_m}{\sum_1^{m-1} \alpha_j}, n_l + 1 \leq n < n_l + 1, \tag{4.33}$$

$$||x_{n_l+1} - x^*||^2 \leq ||x_1 - x^*||^2 - \sum_1^{n} \frac{\alpha_m}{\sum_1^{m-1} \alpha_j} \leq ||x_1 - x^*||^2, 1 \leq n \leq n_1 - 1, \tag{4.34}$$

$$1 \leq n_1 \leq s_{max} = \max\{s : \sum_1^{s} \frac{\alpha_m}{\sum_1^{m} \alpha_j} \leq ||x_1 - x^*||^2\}. \tag{4.35}$$

Proof Since we have by hypothesis that A is bounded, the proof follows as in the proof of Theorem 4.1 without the use of Lemma 4.2.2.

If T is a *self–map* and $0 \leq \alpha_n < 1$, the use of the operator P will not be necessary. To present our next Theorem, we shall need the following Lemma.

Lemma 4.3.1 Let $\{\lambda_k\}$ and $\{\gamma_k\}$ be sequences of nonnegative numbers and $\{\alpha_k\}$ a sequence of positive numbers satisfying the conditions $\sum_1^\infty \alpha_n = \infty$ and $\frac{\gamma_n}{\alpha_n} \to 0$, as $n \to \infty$. Let the recursive inequality

$$\lambda_{n+1} \leq \lambda_n - 2\alpha_n\phi(\lambda_{n+1}) + \gamma_n, n = 1, 2, ..., \tag{4.36}$$

be given where $\phi(\lambda)$ is a continuous and nondecreasing function from $\Re^+$ to $\Re^+$ such that it is positive on $\Re^+ \setminus \{0\}$, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$. Then $\lambda_n \to 0$, as $n \to \infty$.

Proof Let $\liminf \lambda_n = a \geq 0$. **Claim:** $a = 0$. Suppose not. Then there exists $N_1 > 0$ such that $\lambda_n \geq \frac{a}{2}$ $\forall n \geq N_1$. Since $\frac{\gamma_n}{\alpha_n} \to 0$, there exists $N_2 > 0$ such that $\frac{\gamma_n}{\alpha_n} \leq \phi(\frac{a}{2})$ which implies $\gamma_n \leq \alpha_n \phi(\frac{a}{2})$ $\forall n \geq N_2$. Then for $n \geq N = \max\{N_1, N_2\}$ we have from (4.36) that

$$\lambda_{n+1} \leq \lambda_n - 2\alpha_n \phi(\frac{a}{2}) + \alpha_n \phi(\frac{a}{2}) = \lambda_n - \alpha_n \phi(\frac{a}{2}), \forall n > N,$$

which implies that $\phi(\frac{a}{2}) \sum \alpha_n < \infty$, a contradiction. Therefore, $a = 0$. Thus, there exists a subsequence $\{\lambda_{n_j}\} \subset \{\lambda_n\}$ such that $\lim \lambda_{n_j} = 0$. For arbitrary $\epsilon > 0$ let $N_3 > 0$ such that $\lambda_{n_j} < \frac{\epsilon}{4}$ $\forall j \geq N_3$ and $N_4 > 0$ such that $\gamma_n \leq 2\alpha_n \phi(\frac{\epsilon}{4})$. Let $N_* := \max\{N_3, N_4\}$ and fix $j_* > N_*$. Then we show that $\lambda_{n_{j_*}+k} < \frac{\epsilon}{4}$ $\forall k \in N \cup \{0\}$. For $k = 0$ the result clearly holds. Suppose it holds for any $k > 0$. Then we show that it holds for $k+1$. Suppose not. Then we have $\lambda_{n_{j_*}+k+1} > \frac{\epsilon}{4}$ and hence from (4.36) we get that

$$\begin{aligned} \frac{\epsilon}{4} < \lambda_{n_{j_*}+k+1} &\leq \lambda_{n_{j_*}+k} - 2\alpha_n \phi(\lambda_{n_{j_*}+k+1}) + 2\alpha_n \phi(\frac{\epsilon}{4}) \\ &\leq \lambda_{n_{j_*}+k} - 2\alpha_{n_{j_*}+k} \phi(\frac{\epsilon}{4}) + 2\alpha_{n_{j_*}+k} \phi(\frac{\epsilon}{4}) = \lambda_{n_{j_*}+k}, \end{aligned}$$

a contradiction. Therefore, $\lambda_{n_{j_*}+k} < \frac{\epsilon}{4}$ $\forall k \in N \cup \{0\}$ and hence $\lambda_n \to 0$ as $n \to \infty$.

Theorem 4.2 Let K be a closed convex subset of a real Banach space. Suppose $T : K \to K$ is a uniformly continuous $d-$weakly contractive and bounded map with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} := x_n - \alpha_n (x_n - Tx_n), \; n \geq 1, \tag{4.37}$$

where $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n \leq d_0$, then, $\{x_n\}$ converges strongly to $x^* \in F(T)$. Moreover, there exist a constant $d > 0$ and a subsequence $\{x_{n_l}\} \subseteq \{x_n\}$ such that

$$||x_{n_l} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + d\overline{\gamma}_{n_l}\Big). \tag{4.38}$$

where $\overline{\gamma}_n := ||(I-T)x_{n+1} - (I-T)x_n||$. Furthermore,

$$||x_{n_l+1} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l+1}\alpha_m} + d\overline{\gamma}_{n_l}\Big) + d\overline{\gamma}_{n_l}, \tag{4.39}$$

$$||x_n - x^*||^2 \leq ||x_{n+1} - x^*||^2 - \sum_{n_l+1}^{n-l} \frac{\alpha_m}{\sum_1^m \alpha_j}, n_l + 1 \leq n < n_l + 1, \tag{4.40}$$

$$||x_{n_l+1} - x^*||^2 \leq ||x_1 - x^*||^2 - \sum_1^n \frac{\alpha_m}{\sum_1^m \alpha_j} \leq ||x_1 - x^*||^2, 1 \leq n \leq n_1 - 1, \tag{4.41}$$

$$1 \leq n_1 \leq s_{max} = \max\{s : \sum_1^s \frac{\alpha_m}{\sum_1^m \alpha_j} \leq ||x_1 - x^*||^2\}. \tag{4.42}$$

Proof Let $x^* \in F(T)$ and let G, r and σ be as in the proof of Theorem 4.1. By uniform continuity of A, for $\epsilon = \frac{\Phi(r^2)}{4r}$, there exists $\delta_* > 0$ such that $||x - y|| < \delta_*$ implies $||Ax - Ay|| < \epsilon$ for all $x, y \in D(T)$. Choose any $0 < \delta \leq \delta_*$ and set $d_0 := min\{1, \frac{\delta}{2\sigma}, \frac{r}{\sigma}\}$.

Claim : $x_n \in G \quad \forall n \geq 1$. We show this by induction. By our choice $x_1 \in G$. Suppose $x_n \in G$. We show that $x_{n+1} \in G$. Suppose not, then $||x_{n+1} - x^*|| > r$ and from (4.37) we have $||x_{n+1} - x^*|| \leq ||x_n - x^*|| + \alpha_n||Ax_n|| \leq r + d_0\sigma \leq 2r$.

Now, by Lemma 4.2.1 and the above estimates we have that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq ||x_n - x^*||^2 - 2\alpha_n\langle Ax_n - Ax^*, j(x_{n+1} - x^*)\rangle \\
&= ||x_n - x^*||^2 - 2\alpha_n\langle Ax_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\quad +2\alpha_n\langle Ax_{n+1} - Ax_n, j(x_{n+1} - x^*)\rangle \\
&\leq ||x_n - x^*||^2 - 2\alpha_n\Phi(||x_{n+1} - x^*||^2) \\
&\quad +2\alpha_n||Ax_{n+1} - Ax_n||||x_{n+1} - x^*|| \\
&\leq ||x_n - x^*||^2 - 2\alpha_n\Phi(||x_{n+1} - x^*||^2)\frac{||x_{n+1} - x^*||}{2r} \\
&\quad +2\alpha_n||Ax_{n+1} - Ax_n||||x_{n+1} - x^*|| \\
&\leq ||x_n - x^*||^2 - 2\alpha_n\Big(\frac{\Phi(r^2)}{2r} - ||Ax_{n+1} - Ax_n||\Big)||x_{n+1} - x^*|| \\
&\leq ||x_n - x^*||^2 - 2\alpha_n(\frac{\Phi(r^2)}{2r} - \epsilon)||x_{n+1} - x^*||
\end{aligned} \tag{4.43}$$

$$\leq \; ||x_n - x^*||^2 - \alpha_n \frac{\Phi(r^2)}{2r} ||x_{n+1} - x^*||, \quad \text{since } \epsilon = \frac{\Phi(r^2)}{4r}$$
$$\leq \; ||x_n - x^*||^2 < r^2,$$

and hence $||x_{n+1} - x^*|| < r$, a contradiction. Therefore, the claim holds. Now we show that $x_n \to x^*$. Since $x_{n+1} - x_n \to 0$, by the uniform continuity of A we have that

$$\overline{\gamma}_{n_l} := ||Ax_{n+1} - Ax_n|| \to 0 \text{ as } n \to \infty.$$

Let $\lambda_n := ||x_n - x^*||^2$, $\gamma_n := 2\alpha_n \sigma \overline{\gamma}_{n_l}$. Then, (4.43) gives

$$\lambda_{n+1} \leq \lambda_n - 2\alpha_n \Phi(\lambda_{n+1}) + \gamma_n.$$

Thus, the conclusion follows from Lemma 4.2.2.

Remark 4.3.1 Theorem 4.1 extends Theorem AG from real Hilbert spaces to the more general real uniformly smooth Banach spaces. Furthermore, the *boundedness* assumption imposed on K in theorem AG is not needed in our more general setting.

Remark 4.3.2 Theorems 4.1 and 4.2 also hold, without any modification in the proofs, for $d-$weakly hemi$-$contractive maps.

Remark 4.3.3 Observe that if T is weakly contractive then clearly it is uniformly continuous and bounded. Moreover, it is $d-$weakly contractive and in Hilbert spaces, $F(T) \neq \emptyset$ (see e.g., [2, 5]). Therefore, theorem 4.2 extends theorem 6.1 of [4] from the class of weakly contractive maps to the class of $d-$weakly contractive maps.

Chapter 5

Iterative Methods for Fixed points of Asymptotically weakly contractive Maps

5.1 Introduction

A subset K of a Banach space E is said to be a *retract* of E if there exists a continuous map $P : E \to K$ such that $Px = x$ for all x in K. In other words, K is a retract of E if the identity restricted to K has a continuous extension to E. Every closed convex set of a normed linear space is a retract (see for e.g. [35]). A map $P : E \to E$ is said to be a *retraction* if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for every y in the range of P.

If $E = H$, a Hilbert space, the *metric projection* $P_K : H \to K$ defined by $P_K x = \bar{x}$ where $\bar{x} \in K$ is the solution to the minimization problem

$$||x - \bar{x}|| = \inf_{\eta \in K} ||x - \eta||.$$

is a nonexpansive retraction from H to K.

Definition 5.1.1 A map T with domain $D(T)$ and range $R(T)$ in E will be called *asymptotically $d-$weakly contractive* if there exists a continuous and nondecreasing function $\Phi : [0,\infty) := \Re^+ \to \Re^+$ such that Φ is positive on $\Re^+ \backslash \{0\}$, $\Phi(0) = 0$, $lim_{t\to\infty}\Phi(t) = \infty$ and for $x,\ y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$|\langle T(PT)^{n-1}x - T(PT)^{n-1}y,\ j(x-y)\rangle| \leq k_n||x-y||^2 - \Phi(||x-y||^2), \forall\ n \geq 1 \qquad (5.1)$$

where $\{k_n\}$ is a real sequence such that $k_n \geq 1, lim_{n\to\infty}k_n = 1$.

Clearly, if T is a *self-map*, then condition (5.1) reduces to the following one:

$$|\langle T^n x - T^n y, j(x-y)\rangle| \leq k_n||x-y||^2 - \Phi(||x-y||^2).$$

Example 5.1.2 Let B denote the unit ball in the Hilbert space l^2 and let $T : B \to l^2$ be defined as follows:

$$T(x_1, x_2, x_3, \ldots) = (0, x_1^2, a_2 x_2, a_3 x_3, \ldots),$$

where $\{a_i\}$ is a sequence of numbers such that $a_2 > 0,\ 0 < a_i < 1,\ i \neq 2$ and $\prod_{i=1}^{\infty} a_i = \frac{1}{3}$. Then for all x, y in B,

$$\begin{aligned}
|\langle T^i x - T^i y, x-y\rangle| &\leq 2\prod_{j=2}^{i} a_j||x-y||^2 \quad \forall i = 2,3,\ldots \\
&= 3\prod_{j=2}^{i} a_j||x-y||^2 - \prod_{j=2}^{i} a_j||x-y||^2 \\
&\leq 3\prod_{j=2}^{i} a_j||x-y||^2 - \frac{1}{3}||x-y||^2.
\end{aligned}$$

Thus, if we define $k_1 = 3a_2,\ k_n = 3\prod_{j=2}^{n} a_j,\ n = 2,3,\ldots$ and $\Phi(t) = \frac{1}{3}t$. Then $\lim_{n\to\infty} k_n = 1$. Hence

$$|\langle T^n x - T^n y, (x-y)\rangle| \leq k_n||x-y||^2 - \Phi(||x-y||^2),$$

implying that T is asymptotically $d-$weakly contractive. Clearly, T is not $d-$weakly contractive. To see this, take for example: $a_2 = 3$, $x = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, \ldots)$ and $y \equiv 0$ to get $|\langle Tx - Ty, x-y\rangle| >$

$||x-y||^2$. This example is a modification of the one given by Goebel and Kirk [39] for a different class of operators.

It is our purpose in this chapter to prove a version of Theorem AG (see chapter 4) in real uniformly smooth Banach spaces. A corollary of our theorem (Theorem 5.1) extends Theorem AG from Hilbert spaces to real uniformly smooth Banach spaces. Furthermore, the *boundedness* requirement on the domain is dispensed with in our more general setting. A related result deals with the approximation of fixed points of uniformly continuous asymptotically $d-$weakly contractive *self-maps* with nonempty fixed point sets, in arbitrary real Banach spaces.

5.2 Preliminaries

The following Lemmas will be needed in the sequel.

Lemma 5.2.1(see, e.g., [31, 34, 65]). *Let E be a normed linear space and J the normalized duality map on E. Then for any given $x, y \in E$, the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \forall j(x+y) \in J(x+y).$$

Lemma 5.2.2 ([3]) *Let $\{\lambda_k\}$ and $\{\gamma_k\}$ be sequences of nonnegative numbers and $\{\alpha_k\}$ be a sequence of positive numbers satisfying the conditions $\sum_1^\infty \alpha_n = \infty$ and $\frac{\gamma_n}{\alpha_n} \to 0$, as $n \to \infty$. Let the recursive inequality*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \phi(\lambda_n) + \gamma_n, n = 1, 2, ..., \tag{5.2}$$

be given where $\phi(\lambda)$ is a continuous and nondecreasing function from $\Re^+$ to $\Re^+$ such that it is positive on $\Re^+ \setminus \{0\}$, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$. Then
(a) $\lambda_n \to 0$, as $n \to \infty$;
(b) there exists a subsequence $\{\lambda_{n_k}\} \subset \{\lambda_n\}$, l=1,2,..., such that

$$\lambda_{n_l} \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + \frac{\gamma_{n_l}}{\alpha_{n_l}}\Big), \tag{5.3}$$

$$\lambda_{n_l+1} \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + \frac{\gamma_{n_l}}{\alpha_{n_l}}\Big) + \gamma_{n_l}, \tag{5.4}$$

$$\lambda_n \leq \lambda_{n_l+1} - \sum_{n_l+1}^{n-l} \frac{\alpha_m}{A_m}, \quad n_l + 1 \leq n < \quad n_l + 1, \quad A_m = \sum_1^{m-1} \alpha_i, \tag{5.5}$$

$$\lambda_{n+1} \leq \lambda_1 - \sum_1^{n} \frac{\alpha_m}{A_m} \leq \lambda_1, \quad 1 \leq n \leq n_1 - 1, \tag{5.6}$$

$$1 \leq n_1 \leq s_{max} = \max\{s : \sum_1^{s} \frac{\alpha_m}{A_m} \leq \lambda_1\}. \tag{5.7}$$

Lemma 5.2.3 ([53]) *Let E be an arbitrary Banach space, $F: D(F) \subseteq E \to E$ be an accretive operator, $x^o \in int\ D(F)$. Then there exist a constant $r_0 > 0$ and a closed ball $S(r_0, x^o) \subset D(F)$ such that for all $x \in D(F)$ the estimate*

$$\langle F(x), J(x - x^o)\rangle \geq r_0||F(x)|| - c_0(||x - x^o|| + r_0),$$

$$c_0 = \sup_{\xi \in S(r_0, x_0)} ||F(\xi)|| < \infty$$

is satisfied.

Let K be a subset of a real Banach space E and let $\{x_n\}$ be any sequence in K satisfying the condition

$$\Phi(||x - y||^2) \geq (k_n - 1)||x - y||^2, \quad x,\ y \in G := \{x_n\} \cup \{x^o\} \tag{5.8}$$

for arbitrary $x^o \in \ int(K)$ and Φ and k_n are as defined earlier. With this we state and prove the following lemma.

Lemma 5.2.4 *Let E be an arbitrary real Banach space and let $T : D(T) \subseteq E \to E$ be an asymptotically $d-$weakly contractive map, and suppose that a fixed point $x^* \in intD(T)$ of T exists. Then the operator $F_n : G \to E$, where $F_n := I - T(PT)^{n-1}$ is bounded on G if condition (5.8) is satisfied.*

Proof For any $x, y \in G$, we have

$$\begin{aligned}
\langle F_n x - F_n y, j(x-y)\rangle &= \langle x-y, j(x-y)\rangle - \langle T(PT)^{n-1}x - T(PT)^{n-1}y\rangle \\
&\geq ||x-y||^2 - |\langle T(PT)^{n-1}x - T(PT)^{n-1}y, j(x-x_y)\rangle| \\
&\geq ||x-y||^2 - \Big[k_n||x-y||^2 - \Phi(||x-y||^2)\Big] \\
&= -(k_n-1)||x-y||) + \Phi(||x-y||^2) \geq 0 \quad \text{(by condition (5.8))}.
\end{aligned}$$

Hence F_n is accretive on G.

Since $x^* \in intD(F_n)$, we have by Lemma (5.2.3) that

$$\langle F_n x, j(x-x^*)\rangle \geq r_0||F_n x|| - c_0(||x-x^*||) + r_0). \tag{5.9}$$

Also, for $j(x-x^*) \in J(x-x^*)$ we have

$$\begin{aligned}
\langle F_n x, j(x-x^*)\rangle &= \langle F_n x - F_n x^*, j(x-x^*)\rangle \\
&= \langle x-x^*, j(x-x^*)\rangle - \langle T(PT)^{n-1}x - T(PT)^{n-1}x^*, j(x-x^*)\rangle \\
&\leq ||x-x^*||^2 + k_n||x-x^*||^2 - \Phi(||x-x^*||^2) \\
&\leq 2k_n||x-x^*||^2. \qquad (5.10)
\end{aligned}$$

From this and (5.9), we get the following estimate:

$$||F_n x|| \leq r_0^{-1}\Big(2k_n||x-x^*||^2 + c_0(||x-x^*|| + r_0)\Big). \tag{5.11}$$

Hence, for each integer $n \geq 1$, F_n is bounded on G.

5.3 Main Results

Theorem 5.1 *Suppose E is a real uniformly smooth Banach space. Suppose K is a closed convex subset of E which is a nonexpansive retract of E with P as the nonexpansive retraction. Suppose $T : K \to E$ is an asymptotically $d-$weakly contractive map, with strictly increasing function Φ and with the sequence $k_n \geq 1$, $\lim k_n = 1$, such that a fixed point $x^* \in int(K)$ of T exists. For arbitrary $x_1 \in K$, suppose the sequence $\{x_n\}$ is defined iteratively by*

$$x_{n+1} := P\Big((1-k_n\alpha_n)x_n + k_n\alpha_n T(PT)^{n-1}x_n\Big), \ n \geq 1, \tag{5.12}$$

where $\lim \alpha_n = 0$ *and* $\sum \alpha_n = \infty$. *Suppose further that* $\{x_n\}$ *satisfies condition (5.8). Let* $F_n = I - T(PT)^{n-1}$. *Then there exists a constant* d_0 *such that if* $0 < \alpha_n \leq d_0$, $\{x_n\}$ *converges strongly to a fixed point* x^* *of* T. *Moreover, there exist a constant* $b > 0$ *and a subsequence* $\{x_{n_l}\} \subseteq \{x_n\}$ *such that*

$$||x_{n_l} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} k_m \alpha_m} + b\overline{\gamma}_{n_l}\Big), \tag{5.13}$$

where $\overline{\gamma}_n := ||j(r_n - x^*) - j(x_n - x^*)||$ *and* $r_n := x_n - k_n \alpha_n F_n x_n$.
Furthermore,

$$||x_{n_l+1} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} k_m \alpha_{n_l}} + b\overline{\gamma}_{n_l}\Big) + bk_{n_l}\alpha_{n_l}\overline{\gamma}_{n_l}, \tag{5.14}$$

$$||x_n - x^*||^2 \leq ||x_{n+1} - x^*||^2 - \sum_{n_l+1}^{n-l} \frac{k_m \alpha_m}{\sum_1^{m-1} k_j \alpha_j}, n_l + 1 \leq n < n_l + 1, \tag{5.15}$$

$$||x_{n_l+1} - x^*||^2 \leq ||x_1 - x^*||^2 - \sum_{1}^{n} \frac{k_m \alpha_m}{\sum_1^{m-1} k_j \alpha_j} \leq ||x_1 - x^*||^2, 1 \leq n \leq n_1 - 1, \tag{5.16}$$

$$1 \leq n_1 \leq s_{max} = \max\{s : \sum_1^s \frac{k_m \alpha_m}{\sum_1^m k_j \alpha_j} \leq ||x_1 - x^*||^2\}. \tag{5.17}$$

Proof We first show that the sequence $\{x_n\}$ is bounded. The iteration sequence (5.12) can be written in the form

$$x_{n+1} = P(x_n - k_n \alpha_n F_n x_n), \quad n \geq 1. \tag{5.18}$$

There exists an $r_1 > 0$ such that $x_1 \in B(r_1, x^*)$. Fix $r \geq r_1$. Let $B = B(r, x^*) \cap K$. Since F_n is bounded on $\{x_n\} \cup \{x^*\}$, there exists a d^* such that

$$||F_n x_n|| \leq d^* \quad \forall x_n \in B. \tag{5.19}$$

Let a d^* satisfying (5.19) be fixed.
Claim $\{x_n\}$ is bounded. Suffices to show that x_n is in B for all $n \geq 1$. By our choice $x_1 \in B$. Assume $x_n \in B$. We show that $x_{n+1} \in B$. Assume for contradiction that $x_{n+1} \notin B$. Then we must have $||x_{n+1} - x^*|| > r$. Let $\epsilon = \frac{\Phi((\frac{r}{2})^2)}{2d^*}$.

As j is uniformly continuous on bounded subsets of E, there exists a $\delta > 0$ such that $x, y \in D(T)$, $||x - y|| < \delta$ implies $||j(x) - j(y)|| < \epsilon$. Let $d_0 := \min\{\frac{1}{m}, \frac{\delta}{2d^*m}, \frac{r}{2d^*m}\}$, where $m = \sup_{n\geq 1} k_n$.

Now,

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||P(x_n - k_n\alpha_n F_n x_n) - Px^*|| \\ &\leq ||x_n - x^* - k_n\alpha_n F_n x_n|| \\ &\leq ||x_n - x^*|| + m\alpha_n||F_n x_n|| \end{aligned}$$

Thus,

$$\begin{aligned} ||x_n - x^*|| &\geq ||x_{n+1} - x^*|| - m\alpha_n||F_n x_n|| \\ &> r - m\alpha_n d^* \geq r - \frac{r}{2} = \frac{r}{2}. \end{aligned}$$

Let $r_n := x_n - k_n\alpha_n F_n x_n$. Now,

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||P\Big(x_n - k_n\alpha_n(x_n - T(PT)^{n-1}x_n)\Big) - Px^*||^2 \\ &\leq ||x_n - x^* - k_n\alpha_n(x_n - T(PT)^{n-1}x_n)||^2 \qquad (5.20) \\ &= ||x_n - x^* - k_n\alpha_n(F_n x_n - F_n x^*)||^2 \\ &\leq ||x_n - x^*||^2 - 2k_n\alpha_n\langle F_n x_n, j(r_n - x^*)\rangle \quad \text{(using Lemma 2.1)} \\ &= ||x_n - x^*||^2 - 2k_n\alpha_n\langle F_n x_n - F_n x^*, j(x_n - x^*)\rangle \\ &\quad -2k_n\alpha_n\langle F_n x_n, j(r_n - x^*) - j(x_n - x^*)\rangle \\ &\leq ||x_n - x^*||^2 - 2k_n\alpha_n\Phi(||x_n - x^*||^2) \\ &\quad +2k_n\alpha_n||F_n x_n||||j(r_n - x^*) - j(x_n - x^*)||. \qquad (5.21) \end{aligned}$$

Since $||r_n - x_n|| \leq k_n\alpha_n||F_n x_n|| \leq m\alpha_n d^* < \delta$ we have that $||j(r_n - x^*) - j(x_n - x^*)|| \leq \frac{\Phi((\frac{r}{2})^2)}{2d^*}$. Thus (5.21) gives the following estimate:

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq ||x_n - x^*||^2 - 2k_n\alpha_n\Phi((\frac{r}{2})^2) + 2k_n\alpha_n d^*\frac{\Phi((\frac{r}{2})^2)}{2d^*} \\ &= ||x_n - x^*||^2 - 2k_n\alpha_n\Phi((\frac{r}{2})^2) + k_n\alpha_n\Phi((\frac{r}{2})^2) \\ &= ||x_n - x^*||^2 - k_n\alpha_n\Phi((\frac{r}{2})^2) \\ &< r^2. \qquad (5.22) \end{aligned}$$

i.e. $||x_{n+1} - x^*|| < r$, a contradiction. Therefore $x_{n+1} \in B$. Thus by induction $\{x_n\}$ is bounded. Next we show that $x_n \to x^*$. Note that $r_n - x_n \to 0$ as $n \to \infty$ and hence by the uniform continuity of j on bounded subsets of E we have that

$$\overline{\gamma}_n := ||j(r_n - x^*) - j(x_n - x^*)|| \to 0 \text{ as } n \to \infty. \tag{5.23}$$

Let $\lambda_n := ||x_n - x^*||^2$ and $\gamma_n := 2k_n\alpha_n d^*\overline{\gamma}_n$, then from inequality (5.21) we obtain that

$$\lambda_{n+1} \leq \lambda_n - 2k_n\alpha_n\Phi(\lambda_n) + \gamma_n. \tag{5.24}$$

The conclusions of the theorem now follow from Lemma AG, completing the proof of the theorem.

If a fixed point x^* is an arbitrary point of $D(T)$ then we have the following corollary.

Corollary 5.2. *Suppose K is a closed convex subset of a real uniformly smooth Banach space. Suppose K is a nonexpansive retract of E with P as the nonexpansive retraction. Suppose $T : K \to E$ is an asymptotically $d-$weakly contractive and bounded map with nonempty fixed point set. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1} := P\Big((1 - k_n\alpha_n)x_n + k_n\alpha_n T(PT)^{n-1}x_n)\Big), \; n \geq 1, \tag{5.25}$$

where $\lim \alpha_n = 0$, $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n \leq d_0$, then, $\{x_n\}$ converges strongly to a fixed point x^* of T. Moreover, there exist a constant $d > 0$ and a subsequence $\{x_{n_l}\} \subseteq \{x_n\}$ such that

$$||x_{n_l} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} k_m\alpha_m} + d\overline{\gamma}_{n_l}\Big). \tag{5.26}$$

where $\overline{\gamma}_n$ is as defined in (5.23).

Furthermore,

$$||x_{n_l+1} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} k_m\alpha_m} + d\overline{\gamma}_{n_l}\Big) + dk_{n_l}\alpha_{n_l}\overline{\gamma}_{n_l}, \tag{5.27}$$

$$||x_n - x^*||^2 \leq ||x_{n+1} - x^*||^2 - \sum_{n_l+1}^{n-l} \frac{k_m\alpha_m}{\sum_1^{m-1} k_j\alpha_j}, n_l + 1 \leq n < n_l + 1, \tag{5.28}$$

$$||x_{n_l+1} - x^*||^2 \leq ||x_1 - x^*||^2 - \sum_1^n \frac{k_m\alpha_m}{\sum_1^{m-1} k_j\alpha_j} \leq ||x_1 - x^*||^2, 1 \leq n \leq n_1 - 1, \tag{5.29}$$

$$1 \leq n_1 \leq s_{max} = \max\{s : \sum_1^s \frac{k_m\alpha_m}{\sum_1^m k_j\alpha_j} \leq ||x_1 - x^*||^2\}. \tag{5.30}$$

Proof Since we have by hypothesis that T (and hence F_n, for each integer $n \geq 1$) is bounded, the proof follows as in the proof of Theorem 5.1 without the use of Lemma 5.2.4.

The following corollary is an explicit generalization of Theorem AG.

Corollary 5.3 *Let $T : K \to E$ be a $d-$weakly contractive map, K a closed convex subset of a real uniformly smooth Banach space E and suppose that a fixed point $x^* \in intK$ of T exists. Then the sequence $\{x_n\}$ defined by*

$$x_1 \in K,\ x_{n+1} = P\left(x_n - \alpha_n(x_n - Tx_n)\right), x = 1, 2, \ldots,$$

where P is the generalized projection onto the set K, $\{\alpha_n\}$ is a sequence of positive numbers such that $\sum \alpha_n = \infty$, $\lim \alpha_n = 0$, converges strongly to x^*.

Moreover, there exist a constant $b > 0$ and a subsequence $\{x_{n_l}\} \subseteq \{x_n\}$ such that

$$||x_{n_l} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + b\overline{\gamma}_{n_l}\Big), \tag{5.31}$$

where $\overline{\gamma}_n := ||j(p_n - x^*) - j(x_n - x^*)||$ and $p_n := x_n - \alpha_n(I - T)x_n$.

Furthermore,

$$||x_{n_l+1} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} \alpha_m} + b\overline{\gamma}_{n_l}\Big) + b\alpha_{n_l}\overline{\gamma}_{n_l}, \tag{5.32}$$

$$||x_n - x^*||^2 \leq ||x_{n+1} - x^*||^2 - \sum_{n_l+1}^{n-l} \frac{\alpha_m}{\sum_1^{m-1} \alpha_j}, n_l + 1 \leq n < n_l + 1, \tag{5.33}$$

$$||x_{n_l+1} - x^*||^2 \leq ||x_1 - x^*||^2 - \sum_1^n \frac{\alpha_m}{\sum_1^{m-1} \alpha_j} \leq ||x_1 - x^*||^2, 1 \leq n \leq n_1 - 1, \tag{5.34}$$

$$1 \leq n_1 \leq s_{max} = \max\{s : \sum_1^s \frac{\alpha_m}{\sum_1^m \alpha_j} \leq ||x_1 - x^*||^2\}. \tag{5.35}$$

Proof By setting $n \equiv 1$, $k_n \equiv 1$ and hence $F_n \equiv F = I - T$ for all integers n in Lemma 2.4, it follows that F is bounded on $\{x_n\} \cup \{x^o\}$. Following the method of proof of Theorem 5.1, we obtain that $\{x_n\}$ is bounded. The rest of the argument now follows exactly as in the proof of Theorem 3.1 to complete proof of Corollary 5.3.

Corollary 5.3 was also obtained independently in [31].

If T is a *self–map*, the use of the operator P will not be necessary. In this case the operator F_n reduces to $F_n = I - T^n$.

The following Lemma which will be needed for our next theorem was proved in [31]. For completeness, we sketch the proof.

Lemma 5.4 [31] *Let $\{\lambda_k\}$ and $\{\gamma_k\}$ be sequences of nonnegative numbers and $\{\alpha_k\}$ a sequence of positive numbers satisfying the conditions $\sum_1^\infty \alpha_n = \infty$ and $\frac{\gamma_n}{\alpha_n} \to 0$, as $n \to \infty$. Let the recursive inequality*

$$\lambda_{n+1} \leq \lambda_n - 2\alpha_n \phi(\lambda_{n+1}) + \gamma_n, n = 1, 2, ..., \tag{5.36}$$

be given where $\phi(\lambda)$ is a continuous and nondecreasing function from $\Re^+$ to $\Re^+$ such that it is positive on $\Re^+ \setminus \{0\}$, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$. Then $\lambda_n \to 0$, as $n \to \infty$.

Proof Let $\liminf \lambda_n = a \geq 0$.

Claim: $a = 0$. Suppose not. Then there exists $N_1 > 0$ such that $\lambda_n \geq \frac{a}{2}$ $\forall n \geq N_1$. Since $\frac{\gamma_n}{\alpha_n} \to 0$, there exists $N_2 > 0$ such that $\frac{\gamma_n}{\alpha_n} \leq \phi(\frac{a}{2})$ which implies $\gamma_n \leq \alpha_n \phi(\frac{a}{2})$ $\forall n \geq N_2$. Then for $n \geq N = \max\{N_1, N_2\}$ we have from (5.36) that

$$\lambda_{n+1} \leq \lambda_n - 2\alpha_n \phi(\frac{a}{2}) + \alpha_n \phi(\frac{a}{2}) = \lambda_n - \alpha_n \phi(\frac{a}{2}), \forall n > N,$$

which implies that $\phi(\frac{a}{2}) \sum \alpha_n < \infty$, a contradiction. Therefore, $a = 0$. Thus, there exists a subsequence $\{\lambda_{n_j}\} \subset \{\lambda_n\}$ such that $\lim \lambda_{n_j} = 0$. For arbitrary $\epsilon > 0$ let $N_3 > 0$ such that $\lambda_{n_j} < \frac{\epsilon}{4}\ \forall j \geq N_3$ and $N_4 > 0$ such that $\gamma_n \leq 2\alpha_n \phi(\frac{\epsilon}{4})$. Let $N_* := \max\{N_3, N_4\}$ and fix $j_* > N_*$. Then we show that $\lambda_{n_{j_*}+k} < \frac{\epsilon}{4}\ \forall k \in N \cup \{0\}$. For $k = 0$ the result clearly holds. Suppose it holds for any $k > 0$. Then we show that it holds for $k+1$. Suppose not. Then we have $\lambda_{n_{j_*}+k+1} > \frac{\epsilon}{4}$ and hence from (5.36) we get that

$$\begin{aligned}\frac{\epsilon}{4} < \lambda_{n_{j_*}+k+1} &\leq \lambda_{n_{j_*}+k} - 2\alpha_n \phi(\lambda_{n_{j_*}+k+1}) + 2\alpha_n \phi(\frac{\epsilon}{4}) \\ &\leq \lambda_{n_{j_*}+k} - 2\alpha_{n_{j_*}+k}\phi(\frac{\epsilon}{4}) + 2\alpha_{n_{j_*}+k}\phi(\frac{\epsilon}{4}) = \lambda_{n_{j_*}+k},\end{aligned}$$

a contradiction. Therefore, $\lambda_{n_{j_*}+k} < \frac{\epsilon}{4}\ \forall k \in N \cup \{0\}$ and hence $\lambda_n \to 0$ as $n \to \infty$.

Theorem 5.5 *Let K be a closed convex subset of a real Banach space. Suppose $T : K \to K$ is a uniformly continuous asymptotically $d-$weakly contractive map with a nonempty fixed point set. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1} := x_n - k_n \alpha_n \Big(x_n - T^n x_n\Big),\ n \geq 1, \tag{5.37}$$

where $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n \leq d_0$, then, $\{x_n\}$ converges strongly to a point x^* of T. Moreover, there exist a constant $b > 0$ and a subsequence $\{x_{n_l}\} \subseteq \{x_n\}$ such that

$$||x_{n_l} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l} k_m \alpha_m} + b\overline{\gamma}_{n_l}\Big), \tag{5.38}$$

where $\overline{\gamma}_n := ||(I-T)x_{n+1} - (I-T)x_n||$. Furthermore,

$$||x_{n_l+1} - x^*||^2 \leq \phi^{-1}\Big(\frac{1}{\sum_1^{n_l+1} k_m \alpha_m} + b\overline{\gamma}_{n_l}\Big) + b\overline{\gamma}_{n_l}, \tag{5.39}$$

$$||x_n - x^*||^2 \leq ||x_{n+1} - x^*||^2 - \sum_{n_l+1}^{n-l} \frac{k_m \alpha_m}{\sum_1^m k_j \alpha_j}, n_l + 1 \leq n < n_l + 1, \tag{5.40}$$

$$||x_{n_l+1} - x^*||^2 \leq ||x_1 - x^*||^2 - \sum_1^n \frac{k_m \alpha_m}{\sum_1^m k_m \alpha_j} \leq ||x_1 - x^*||^2, 1 \leq n \leq n_1 - 1, \tag{5.41}$$

$$1 \leq n_1 \leq s_{max} = \max\{s : \sum_1^s \frac{k_m \alpha_m}{\sum_1^m k_j \alpha_j} \leq ||x_1 - x^*||^2\}. \tag{5.42}$$

Proof We first show that the operator $F_n : K \to K$, where $F_n := I - T^n$ is uniformly continuous. Suffices to show that T^n is uniformly continuous. Given $\epsilon > 0$, there exists $\delta_1 > 0$ such that $||x - y|| < \delta_1$ implies $||Tx - Ty|| < \epsilon$. There exists $\delta_2 > 0$ such that $||T^{n-1}x - T^{n-1}y|| < \delta_2$ implies $||T^n x - T^n y|| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $||x - y|| < \delta \Rightarrow ||T^n x - T^n y|| < \epsilon$. Thus T^n and hence F_n is uniformly continuous for each integer $n \geq 1$.

Let x^* be a fixed point of T and let B, r and d^* be as in the proof of theorem 5.1.

Claim : $x_n \in B \quad \forall n \geq 1$. By our choice $x_1 \in B$. Assume $x_n \in B$. We show that $x_{n+1} \in B$. Suppose not, then $||x_{n+1} - x^*|| > r$. By the uniform continuity of F, for $\epsilon = \frac{\Phi(r^2)}{4r}$, there exists $\delta_1 > 0$ such that $||x - y|| < \delta_1$ implies $||F_n x - F_n y|| < \epsilon$ for all $x, y \in D(T)$. Fix δ, $0 < \delta \leq \delta_1$ and set $d_0 := min\{\frac{1}{m}, \frac{r}{d^* m}, \frac{\delta}{2d^* m}\}$.

Now, $||x_{n+1} - x^*|| \leq ||x_n - x^*|| + m\alpha_n||F_n x_n|| \leq r + d_0 d^* \leq 2r$.

Using this and Lemma 5.1 we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq ||x_n - x^*||^2 - 2k_n\alpha_n\langle F_n x_n, j(x_{n+1} - x^*)\rangle \\
&= ||x_n - x^*||^2 - 2k_n\alpha_n\langle F_n x_{n+1} - F_n x^*, j(x_{n+1} - x^*)\rangle \\
&\quad +2k_n\alpha_n\langle F_n x_{n+1} - F_n x_n, j(x_{n+1} - x^*)\rangle \\
&\leq ||x_n - x^*||^2 - 2k_n\alpha_n\Phi(||x_{n+1} - x^*||^2) \\
&\quad +2k_n\alpha_n||F_n x_{n+1} - F_n x_n||||x_{n+1} - x^*|| \qquad (5.43)
\end{aligned}$$

From inequality (5.43) we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq ||x_n - x^*||^2 - 2k_n\alpha_n\Phi(r^2) + 2k_n\alpha_n\frac{\Phi(r^2)}{2} \\
&\leq ||x_n - x^*||^2 - k_n\alpha_n\Phi(r^2) < r^2 \qquad (5.44)
\end{aligned}$$

and hence $||x_{n+1} - x^*|| < r$, a contradiction. Thus $\{x_n\}$ is bounded. Now we show that $x_n \to x^*$.

Since $x_{n+1} - x_n \to 0$, by the uniform continuity of F we have that

$$\overline{\gamma}_n := ||F_n x_{n+1} - F_n x_n|| \to 0 \text{ as } n \to \infty.$$

Let $\lambda_n := ||x_n - x^*||^2$, $\gamma_n := 2k_n \alpha_n \overline{\gamma}_n$. Then, (5.43) gives

$$\lambda_{n+1} \leq \lambda_n - 2k_n \alpha_n \Phi(\lambda_{n+1}) + \gamma_n.$$

Thus, the conclusion follows from Lemma 5.3.3.

Remark 5.6 Prototype of Φ satisfying condition (5.8) is $\Phi(t) = t$.

Remark 5.7 Corollary 5.3 extends Theorem AG from real Hilbert spaces to the more general real uniformly smooth Banach spaces. Furthermore, the *boundedness* assumption imposed on K in theorem AG is not needed in our more general setting.

Chapter 6

Mathematical Modelling of Drug Resistant Malaria Parasites and Vector Populations

6.1 Introduction

Recall the following fundamental mathematical model for malaria, due to Ross-Macdonald. This model describes the basic features of the interaction between infected humans (y) and infected mosquito vectors (q). The model is defined as follows:

$$\begin{aligned} \dot{y} &= \alpha q(1-y) - ry, \\ \dot{q} &= \beta y(1-q) - \mu q, \end{aligned}$$

where $\alpha = b\beta\frac{M}{N}$, β, r, μ are some constants defined as follows (see for e.g. [12]):

N is the size of the human population;

M is the size of the female mosquito population;

$\frac{M}{N}$ is the number of female mosquitoes per human host;

β is the rate of biting on man by a single mosquito (number of bites per unit time);

b is the proportion of infected bites on man that produce an infection;

r is the per capita rate of recovery for humans ($\frac{1}{r}$ is the average duration of infection in the human host);

μ is the per capita mortality rate for mosquitoes ($\frac{1}{\mu}$ is the average lifetime of a mosquito);

(the constants have their usual meanings, see our General Introduction).

In this chapter, we describe the phenomenon of drug resistant malaria infection in a hyperendemic region by a system of ordinary differential equations models. Different situations on both vectors and humans are described. The first presentations deal with single population each for both humans and vectors (with the human population fixed). These models are without resistant strains. The second group of models deal with both resistant and sensitive strains. Our focus in analysing the models is on establishing a unique positive asymptotic equilibrium for each of them. It is shown that (under suitable conditions in most cases) the equilibrium points are (locally) asymptotically stable. The biological significance of each of these points emerges as a by-product. Throughout this paper, we shall adopt the following notations (where applicable):

N denotes the total human population size which is assumed to be fixed,

x denotes susceptibles (the number of people that are uninfected),

y denotes infectives (the number of people that are severely infected).

Where y, and Y occur simultaneously, we designate y as the number of individuals infected with parasites that are sensitive to drugs and Y, the number of individuals that are infected with resistant parasites with or without the sensitive parasites.

6.2 Simple host–vector model

We start by assuming that all parasites are sensitive to the drug (chloroquine or other 4-aminoquinoline derivatives). This can be a case of some acute malaria caused by single or mixed infections of P.

falciparum and P. vivax. Let q be the population of infected mosquitoes, r the cure rate, and c, the unit contact rate between infected mosquitoes and susceptibles. We have (using x+y=N),

$$\dot{y} = cxq - ry = c(N - y)q - ry.$$

If all infectives are treated initially and respond to treatment, then there are no gametocytes. This is also on the assumption that all infectives become 'immediately' symptomatic. This kind of infection is most common in infants. So, we could assume that this model depicts infant infection, where the mosquitoes become infected from infected adult humans. In this case, we assume that the infected mosquitoes grow by an influx v. Then , $\dot{q} = v - \delta q$ (where δ is the death rate of the mosquitoes). So, we have the following nonlinear system:

$$\dot{y} = -ry + cNq - cyq, \tag{6.1}$$

$$\dot{q} = v - \delta q, \tag{6.2}$$

The equilibrium point of this system is given by

$$(\hat{y}, \hat{q}) = \left(\frac{vcN}{vc + r\delta}, \frac{v}{\delta} \right).$$

Differential equation (6.2) is easily solved for q, giving

$$q(t) = \frac{v}{\delta}\left(1 - e^{-\delta t}\right) + q(0)e^{-\delta t};$$

which approaches $\frac{v}{\delta}$ as $t \to \infty$. With this limit for q,

$$\dot{y}(t) = -(r + \frac{cv}{\delta})y + \frac{cNv}{\delta},$$

which solves to give

$$y(t) = \frac{vcN}{vc + r\delta}\left(1 - e^{\frac{-t(r\delta + cv)}{\delta}}\right) + y(0)e^{\frac{-t(r\delta + cv)}{\delta}}.$$

$y(t)$ approaches $\frac{vcN}{vc+r\delta}$ as $t \to \infty$; hence the equilibrium point is locally asymptotically stable.

The following proposition provides a rationale for this approach.

PROPOSITION PS ([55]).

Consider the system

$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2)\\ \dot{x}_2 &= g(x_2)\end{aligned}$$

with $f(.,.)$ and $g(.)$ continuous throughout a compact subset E of $\mathcal{R}^2$. Define $P(E)$ as the projection of E onto the x_2 axis. Assume

1. E is positively invariant for the system,

   and

2. $x_2 = \hat{x}_2$ is an equilibrium point that is globally asymptotically stable on a subset A of $P(E)$.

Then every trajectory starting in $A' = \{(x_1, x_2)\} : \; x_2 \in A\}$ tends asymptotically to a point of the form $(\hat{x}_1, \hat{x}_2)$, where $\hat{x}_1$ is an equilibrium of

$$\dot{x}_1 = f(x_1, \hat{x}_2) \tag{*}$$

Proof

By assumption (2) the ω limit set of every trajectory starting in A' must lie on the line $x_2 = \bar{x}_2$. By Poincaré-Bendixon theory and the continuity of the functions f and g, the ω limit set must consist of a single point (see for e.g. [40], p. 155), say, $(\bar{x}_1, \bar{x}_2)$. Clearly this point must be an equilibrium of the system. Hence $\bar{x}_1$ must also be an equilibrium of (*). This completes the proof.

Now if we take $E = \{(y, q) | \; 0 \leq y \leq N, \; 0 \leq q \leq (\frac{2v}{\delta})$, then E is a compact subset of $\mathcal{R}^2$ and is positively invariant for the system (6.1) and (6.2).

It is seen from this model that the vector population, and infection growth are controlled more or less by the parameter δ. As $\delta \to 0$, q becomes unbounded and $y(t) \to N$.

A more appropriate situation (in considering the entire human population) within a given region is to assume that the population of infected mosquitoes grow by a factor proportional to the number of infectives, so that $\dot{q} = \nu y - \delta q$. Thus, we consider the following

$$\begin{aligned} \dot{y} &= -ry + cNq - cyq, \qquad (6.3) \\ \dot{q} &= \nu y - \delta q, \qquad (6.4) \end{aligned}$$

The system (6.3) and (6.4) has two equilibrium points given by $(y, q) = (0, 0)$ or $(\frac{-r\delta+\nu cN}{\nu c}, \frac{-r\delta+\nu cN}{c\delta})$. Both eigenvalues of the associated matrix of the linearized system near the origin have negative real parts provided $N < \frac{r\delta}{\nu c}$. Thus, stability of the origin is dependent on the magnitudes of the parameters. Let $X = (y, q)^T$, $\dot{X} = f(X)$, and $X_0 = (y_0, q_0)^T$, where

$$y_0 = \frac{-r\delta + \nu cN}{\nu c}, \quad q_0 = \frac{-r\delta + \nu cN}{c\delta}. \qquad (6.5)$$

Then,

$$\nabla f(X_0) = \begin{bmatrix} -r - cq_0 & cN - cy_0 \\ \nu & -\delta \end{bmatrix}. \qquad (6.6)$$

It is easy to see by Routh-Hurwitz conditions that both eigenvalues of the matrix in (6.6) have negative real parts provided that N $< y_0 + \frac{\delta(r+cq_0)}{\nu c}$. This inequality simplifies to $0 < y_0$, (since $\frac{\delta(r+cq_0)}{\nu c} = N$). Now, $N < \frac{r\delta}{\nu c}$ implies $\nu cN < r\delta$, which implies $y_0 < 0$. Thus, the point (y_0, q_0) is not feasible if $N < \frac{r\delta}{\nu c}$. All these prove the following theorem, which we state formally as follows:

THEOREM 6.1

The origin is a locally asymptotically stable equilibrium point of the system (6.3) and (6.4) provided that $N < \frac{r\delta}{\nu c}$ when it is the only equilibrium point. If $N > \frac{r\delta}{\nu c}$ then the origin is unstable and there is a unique endemic equilibrium (y_0, q_0) which is locally stable.

6.3 Resistant parasites

We consider infections with both sensitive and resistant strains of the parasite. Let r and R be the cure rates for the sensitive and resistant infectives respectively; q and Q the populations of mosquitoes carrying sensitive and resistant parasites respectively. Note that it is possible for some mosquitoes of the class Q to be carrying both strains of the parasite. If a human susceptible is infected with both strains of the parasite, we simply call it resistant infection since the sensitive infection is absorbed in the resistant. Let u be the probability that a susceptible coming into infective contact with the mosquito class Q is infected with a resistant strain, given that the susceptible is infected and $1-u$ the complementary probability, that is the probability that the susceptible is infected with a sensitive strain of the parasite. As in the single class sensitive case, we first assume that the population of infected mosquitoes grow by a certain influx. We then have the following mosquito–sensitive–resistant model

$$\dot{y} = cxq + c(1-u)xQ - cuyQ - ry, \tag{6.7}$$
$$\dot{Y} = cuxQ + cuyQ - RY, \tag{6.8}$$
$$\dot{q} = v - \delta q, \tag{6.9}$$
$$\dot{Q} = V - \Delta Q, \tag{6.10}$$
$$\dot{x} = -cx(y+Y) + ry + RY, \tag{6.11}$$

where as before c is the unit contact rate between susceptibles and infected mosquitoes with which new cases occur, δ and Δ are the death or removal rates of sensitive and resistant type mosquitoes respectively. The contribution $cuyQ$ is a case of superinfection. Such a phenomenon has been known to occur in reality (see for e.g. [12, 14]). Eliminating x from (6.7) and (6.8) (using $x+y+Y=N$) we get:

$$\dot{y} = -ry + cNq + cN(1-u)Q - cyq - cYq - cyQ - c(1-u)YQ, \tag{6.12}$$
$$\dot{Y} = -RY + cuNQ - cuYQ, \tag{6.13}$$
$$\dot{q} = v - \delta q, \tag{6.14}$$

$$\dot{Q} = V - \Delta Q, \tag{6.15}$$

Equations (6.13) and (6.15) can be identified with equations (6.1) and (6.2) with cu as the contact parameter. It follows that the equation for resistant infection in the double strain model can be identified with that of the sensitive infection in the single strain model. Equations (6.14) and (6.15) have solutions

$$q(t) = \frac{v}{\delta} + \left(q(0) - \frac{v}{\delta}\right) e^{-\delta t};$$
$$Q(t) = \frac{V}{\Delta} + \left(Q(0) - \frac{V}{\Delta}\right) e^{-\Delta t}.$$

To proceed with the stability analysis of the system (6.12)–(6.15) we shall prove a more general version of Proposition PS with $(x_1, x_2) \in \mathcal{R}^4$ and $x_2 \in \mathcal{R}^3$. Clearly $q \to \frac{v}{\delta}$, $\quad Q \to \frac{V}{\Delta}$ as $t \to \infty$.

By application of Proposition PS with $x_2 \in \mathcal{R}^2$ to the equations (6.13) and (6.15) we deduce $Y \to \frac{VcuN}{Vcu+R\Delta}$. Hence equation (6.13) has the form

$$\dot{y} = -(r + cq + cQ)y + cNq + cN(1-u)Q - cYq - c(1-u)YQ$$

or

$$\dot{y} = -A(t)y + B(t)$$

where $A(t) \to r + \frac{cv}{\delta} + \frac{cV}{\Delta} > 0$ as $t \to \infty$,
$B(t) \to B = \frac{cvN}{\delta} + \frac{cVN}{\Delta} - \frac{cuVN}{\Delta} - \frac{c^2uvVN}{\delta(Vcu+R\Delta)} - \frac{c^2(1-u)V^2uN}{\Delta(Vcu+R\Delta)}$ as $t \to \infty$.
Also $B > 0$ since,

$$\begin{aligned} B &= \frac{cvN}{\delta} - \frac{c^2uvVN}{\delta(Vcu+R\Delta)} + \frac{cVN}{\Delta} - \frac{cuVN}{\Delta} - \frac{c^2(1-u)V^2uN}{\Delta(Vcu+R\Delta)} \\ &= \frac{cvN}{\delta}\left(1 - \frac{Vcu}{(Vcu+R\Delta)}\right) + \frac{cVN}{\Delta}(1-u)\left(1 - \frac{Vcu}{(Vcu+R\Delta)}\right) \\ &> 0. \end{aligned}$$

Hence $B(t) \to B > 0$ as $t \to \infty$.
As $\frac{B+x}{A-x} \to \frac{B}{A}$ as $x \to 0$, given $\epsilon > 0$, $\exists \eta > 0$ s.t. if $|x| \leq \eta \qquad \frac{B}{A} - \frac{1}{2}\epsilon < \frac{B+x}{A-x} < \frac{B}{A} + \frac{1}{2}\epsilon$

$\exists t_o$ s.t. for $t \geq t_o$

$$
\begin{aligned}
A + \eta &\geq A(t) \geq A - \eta > 0 \\
B + \eta &\geq B(t) \geq B - \eta > 0
\end{aligned}
$$

w.l.o.g. let $\eta < \frac{1}{2}\min(A, B)$, $\quad \epsilon < 1$

$$-(A+\eta)y - (B-\eta) \leq \dot{y} \leq -(A-\eta)y + (B+\eta)$$

As

$$
\begin{aligned}
\dot{y} &\leq -(A-\eta)y + (B+\eta) \quad \text{for } t \geq t_o \\
&= -(A-\eta)\left\{y - \frac{B+\eta}{A-\eta}\right\}
\end{aligned}
$$

$$\dot{\zeta} \leq -(A-\eta)\zeta \quad \text{for} \quad t \geq t_o \quad \text{where} \quad \zeta = y - \frac{B+\eta}{A-\eta}.$$

So

$$
\begin{aligned}
\zeta(t) &\leq \zeta(t_o)e^{-(A-\eta)(t-t_o)} \\
&\leq \left(N - \frac{B}{A} + \frac{1}{2}\epsilon\right)e^{-(A-\eta)(t-t_o)} \\
&\leq \left(N - \frac{B}{A} + \frac{1}{2}\right)e^{-\frac{1}{2}A(t-t_o)} \\
&\leq \frac{1}{2}\epsilon \quad \text{if} \quad t > t_1 \quad \text{where} \quad t_1 \quad \text{does not depend on} \quad \epsilon \\
y &\leq \frac{B+\eta}{A-\eta} + \frac{1}{2}\epsilon \leq \frac{B}{A} + \epsilon \quad \text{if} \quad t > t_1.
\end{aligned}
$$

Similarly $\exists t_2 > t_o$ s.t. if $t > t_2$ $\; y \geq \frac{B}{A} - \epsilon$. So $y \to \frac{B}{A}$ as $t \to \infty$ which is the required result.

Thus, we conclude that as in the single strain case, the double strain counterpart has a unique positive asymptotic equilibrium.

We now consider a two strain model equivalence of equations (6.3) and (6.4). Since the sensitive vectors grow by factors proportional to both itself and the resistant type, equations (6.14) and (6.15) are replaced by

$$\dot{q} = \alpha_1 y + \alpha_2 Y - \delta q, \tag{6.16}$$

$$\dot{Q} = \gamma Y - \Delta Q. \tag{6.17}$$

The system (6.12), (6.13), (6.16), and (6.17) has four equilibrium points in which the origin is one. By considering the linearized system near the origin, it is easy to see that the origin is locally asymptotically stable if

$$\begin{array}{rrcl} & N & < & min\left\{\frac{r\delta}{\alpha_1 c}, \frac{R\Delta}{cu\gamma}\right\} \\ \text{and unstable if} & N & > & min\left\{\frac{r\delta}{\alpha_1 c}, \frac{R\Delta}{cu\gamma}\right\}. \end{array}$$

The other equilibrium points are
$P_1(y_1, Y_1, q_1, Q_1) = (\frac{Nc\alpha_1 - r\delta}{c\alpha_1}, 0, \frac{Nc\alpha_1 - r\delta}{c\delta}, 0)$, $P_2(y_2, Y_2, q_2, Q_2) = (y_2, \frac{cu\gamma N - R\Delta}{cu\gamma}, q_2, \frac{cu\gamma N - R\Delta}{cu\Delta})$, $P_3(y_3, Y_3, q_3, Q_3) = (y_3, \frac{cu\gamma N - R\Delta}{cu\gamma}, q_3, \frac{cu\gamma N - R\Delta}{cu\Delta})$; and each of (y_2, q_2), (y_3, q_3) satisfies some quadratic equations of the form

$$EX^2 + FX + G = 0, \tag{6.18}$$

where E, F, and G are some functions of the parameters. Let $P = (y, Y, q, Q)^T$ be any point in $\mathcal{R}^4$, and let $P_i = (y_i, Y_i, q_i, Q_i)^T \quad (i = 0, 1, 2, 3)$ be any of the equilibrium points. If $\dot{P} = g(P_i)$, then

$$\nabla g(P_i) = \begin{bmatrix} -r - cq_i - cQ_i & -cq_i - c(1-u)Q_i & cN - cy_i - cY_i & cN(1-u) - cy_i - c(1-u)Y_i \\ 0 & -R - cuQ_i & 0 & cuN - cuY_i \\ \alpha_1 & \alpha_2 & -\delta & 0 \\ 0 & \gamma & 0 & -\Delta \end{bmatrix} \tag{6.19}$$

By a classical theorem of Lyapunov, P_i is asymptotically stable if and only if the real parts of the eigenvalues of the matrix in (6.19) are negative. It is easy to see that all the eigenvalues of the matrix (6.19) are negative provided

$$N < min\{Y_i + \frac{\Delta Q_i}{\gamma} + \frac{\Delta R}{cu\gamma}, \; y_i + Y_i + \frac{\delta q_i}{\alpha_1} + \frac{\delta Q_i}{\alpha_1} + \frac{r\delta}{c\alpha_1}\}. \tag{6.20}$$

Now we consider all the possible cases:

(i) If $N < \min\left\{\frac{r\delta}{\alpha_1 c}, \frac{R\Delta}{cu\gamma}\right\}$

then the origin is locally asymptotically stable and none of the points P_1, P_2 and P_3 are biologically feasible.

(ii) If $\frac{R\Delta}{cu\gamma} > N > \frac{r\delta}{\alpha_1 c}$

then the origin is biologically feasible and unstable. P_1 is feasible and

$$Y_1 + \frac{\Delta Q_1}{\gamma} + \frac{\Delta R}{cu\gamma} = \frac{\Delta R}{cu\gamma} > N,$$

$$\begin{aligned} y_1 + Y_1 + \frac{\delta q_1}{\alpha_1} + \frac{\delta Q_1}{\alpha_1} + \frac{r\delta}{c\alpha_1} &= 2\left(\frac{Nc\alpha_1 - r\delta}{c\alpha_1}\right) + \frac{r\delta}{c\alpha_1} \\ &= \frac{Nc\alpha_1 - r\delta}{c\alpha_1} + N > N. \end{aligned}$$

So P_1 is locally asymptotically stable. Neither P_2 nor P_3 is biologically feasible in this case.

(iii) If $\frac{r\delta}{\alpha_1 c} > N > \frac{R\Delta}{cu\gamma}$

then the origin is unstable and P_1 is not biologically feasible. (y_2, q_2) and (y_3, q_3) are the roots (y, q) of the equations

$$\begin{aligned} -ry + cNq + cN(1-u)Q - cyq - cYq - cyQ - c(1-u)YQ &= 0 \qquad (i) \\ \alpha_1 y + \alpha_2 Y - \delta q &= 0 \qquad (ii) \end{aligned}$$

where $Y = Y_2 = Y_3$ and $Q = Q_2 = Q_3$.

Substituting (ii) into (i) we have:

$$-(r + cq + cQ)\left(\frac{\delta q - \alpha_2 Y}{\alpha_1}\right) + cNq - cYq + cN(1-u)Q - cY(1-u)Q = 0.$$

This is a quadratic in q given by

$$\begin{aligned} &- \frac{c\delta}{\alpha_1} q^2 \\ &+ \left[-(r + cQ)\frac{\delta}{\alpha_1} + \frac{c\alpha_2 Y}{\alpha_1} + c(N - Y)\right] q \\ &+ \left[(r + cQ)\frac{\alpha_2 Y}{\alpha_1} + c(N - Y)(1-u)Q\right] = 0 \end{aligned}$$

say $X_o q^2 + X_1 q + X_2 = 0$. As $X_o < 0$ and $X_2 > 0$ the product of the roots $q_2 q_3 < 0$, hence exactly one of q_2 and q_3 is positive. Suppose that

$$q_2 = \frac{-X_1 - \sqrt{X_1^2 - 4X_o X_2}}{2X_o} > 0$$
$$q_3 = \frac{-X_1 + \sqrt{X_1^2 - 4X_o X_2}}{2X_o} < 0.$$

If $h(q) = X_o q^2 + X_1 q + X_2$ then $h\left(\frac{\alpha_2 Y}{\delta}\right) > 0$, so $q_2 > \frac{\alpha_2 Y}{\delta}$.

So $y_2 = \frac{\delta q_2}{\alpha_1} - \frac{\alpha_2 Y}{\alpha_1} > 0$. Hence the equilibrium point P_2 is biologically feasible whereas P_3 is biologically infeasible.

Moreover

$$\begin{aligned}
Y_2 + \frac{\Delta Q_2}{\gamma} + \frac{\Delta R}{cu\gamma} &= 2\left(\frac{cu\gamma N - R\Delta}{cu\gamma}\right) + \frac{R\Delta}{cu\gamma} \\
&= \frac{cu\gamma N - R\Delta}{cu\gamma} + N > N \\
y_2 + Y_2 + \frac{\delta q_2}{\alpha_1}\frac{\delta Q_2}{\alpha_1} + \frac{r\delta}{c\alpha_1} &= \frac{2\delta q_2}{\alpha_1} - \frac{\alpha_2 Y_2}{\alpha_1} + \frac{r\delta}{c\alpha_1} + Y_2 + \frac{\delta Q_2}{\alpha_1}.
\end{aligned}$$

Now $q_2 > -\frac{X_1}{2X_o}$ so $-2X_o q_2 > X_1$.

$$\begin{aligned}
\frac{2\delta q_2}{\alpha_1} &> \frac{\alpha_2 Y_2}{\alpha_1} + (N - Y_2) - (r + cQ_2)\frac{\delta}{\alpha_1 c}, \\
\frac{2\delta q_2}{\alpha_1} &- \frac{\alpha_2 Y_2}{\alpha_1} + \frac{r\delta}{\alpha_1 c} + Y_2 + \frac{cQ_2}{\alpha_1} > N.
\end{aligned}$$

So P_2 is locally asymptotically stable.

(iv) If $N > \max\left\{\frac{r\delta}{\alpha_1 c}, \frac{R\Delta}{cu\gamma}\right\}$

then the origin, P_1 and P_2 are biologically feasible. The origin is unstable. Also

$$Y_1 + \frac{\Delta Q_1}{\gamma} + \frac{\Delta R}{cu\gamma} = \frac{\Delta R}{cu\gamma} < N.$$

So P_1 is unstable. P_2 is locally stable (as in (iii)). P_3 is infeasible (as in (iii)).

We now state the foregoing analysis in a theorem as follows:

THEOREM 6.2

The system (6.12), (6.13), (6.16), and (6.17) has the following stability analysis.

(i) If $N < \min\left\{\frac{r\delta}{\alpha_1 c}, \frac{R\Delta}{cu\gamma}\right\}$
then the origin is locally asymptotically stable and none of P_1, P_2 and P_3 are biologically feasible.

(ii) If $\frac{r\Delta}{cu\gamma} > N > \frac{r\delta}{\alpha_1 c}$
then the origin is unstable, P_1 is biologically feasible and asymptotically stable and P_2 and P_3 are biologically infeasble.

(iii) If $\frac{r\delta}{\alpha_1 c} > N > \frac{R\Delta}{cu\gamma}$
then the origin is unstable and P_1 is not biologically feasible, P_2 is biologically feasible and asymptotically stable and P_3 is infeasible.

(iv) If $N > \max\left\{\frac{r\delta}{\alpha_1 c}, \frac{R\Delta}{cu\gamma}\right\}$
then the origin, P_1 and P_2 are all biologically feasible. P_2 is locally asymptotically stable whereas the origin and P_1 are both unstable. P_3 is infeasible.

6.4 Conclusion

This paper discusses malaria infection in an endemic region. The first two models treat sensitive infections only. These models do not assume fixity of vector populations, thus making them more of reality than the fundamental model of Ross-Macdonald and most of its products.

The first and third models are typical of a situation in which there is a sudden outbreak of infections. In this case, infected vectors arrive at a constant rate from a region that is already endemic. Each of the models has a unique positive endemic state. The first model indicates that nearly everybody could be infected with the parasite if the death rate of vectors is small. This is seen from the fact that the endemic state of y, $\hat{y} \to N$ as $\delta \to 0$. Similarly, in the third model, resistant infectives persist and nearly everybody could be infected with resistant parasites if the death rate is small. This is seen from the fact that the endemic state of Y, $\hat{Y} \to N$ as $\Delta \to 0$. Thus, since resistant infectives are less sensitive to the drugs, major health efforts should be geared toward destroying the parasites.

The second and fourth models are typical of most African countries. In this case, the growth of infected vectors is proportional to the number of infectives. Under suitable conditions in each case, the origin is locally asymptotically stable. The fourth model, in particular, describes more accurately the current trend in malaria infection, especially in Nigeria for instance, where strains that are resistant to virtually all known drugs have emerged. Since efforts should be directed towards eliminating resistant parasites (and hence resistant infection), conditions (i) and (ii) of theorem 2 are important eye-opener to health management board. They indicate when the origin or P_1 is locally asymptotically stable. On the other hands, the conditions (iii) and (iv) of theorem 2 is not favourable to the elimination of resistant infection, since in these cases, the point P_2 leads to a positive endemic state for all infections.

Finally, each of the models can be adapted to describe the spread of other infectious diseases whose biology is similar to that of malaria.

Chapter 7

Some Malaria Models Treating both Sensitive and Resistant Strains in Single and Multigroup Populations

7.1 Introduction

In this chapter we continue our study of drug resistant strains of malaria parasites. We consider a nonlinear model treating both sensitive and resistant strains, first in a single population and secondly in a multigroup population. We assume in each case that all infected individuals receive initial treatment. This consists in the administration of chloroquine or other 4-aminoquinoline derivatives. Each of our models has only two equilibrium points–a trivial and a nontrivial equilibria. Our observation is that resistant parasites constitute the major health problems since sensitive parasites are destroyed by drug action. It then follows that under this setting, sensitive parasites grow by an influx, metamorphosis, and/or by evolution. However our model does not recognise infected vector growth by influx. Our models reveal that either both infections die out simultaneously or an endemic state is reached for both of them.

Throughout this chapter, we shall adopt the following notations (where applicable):

N denotes the total human population size which is assumed to be fixed,
x denotes susceptibles the number of people that are uninfected),
y denotes infectives (the number of people that are severely infected),
z denotes the immunes (the number with mild, asymptomatic infections).
Where y, and Y occur simultaneously, we designate y as the number of individuals infected with parasites that are sensitive to drugs and Y, the number of individuals that are infected with resistant parasites with or without the sensitive parasites.

7.2 Single population

We assume that all infected individuals receive certain popular drugs such as chloroquine or other 4-aminoquinoline derivatives as their initial and only treatment. We consider infections with both drug sensitive and resistant strains of the parasite. Let r and R be the cure rates for the sensitive and resistant infecteds respectively; q and Q the populations of mosquitoes carrying sensitive and resistant parasites respectively. Note that it is possible for some mosquitoes of the class Q to be carrying both strains of the parasite. If a human susceptible is infected with both strains of the parasite, we simply call it resistant infection since the sensitive infection is absorbed in the resistant. Let u be the probability that a susceptible coming into infective contact with the mosquito class Q is infected with a resistant strain, given that the susceptible is infected and $1-u$ the complementary probability, that is the probability that the susceptible is infected with a sensitive strain of the parasite. Since infecteds are treated initially, sensitive infecteds return directly to the susceptible group while resistant infecteds gradually acquire immunity and pass onto the immune class. Thus, sensitive infecteds cannot (possibly) produce *gametocytes* (matured strains) that are responsible for the transmission of the disease. Hence, the class of (partial) immunes can only arise from the class of resistant infecteds. Thus, we have the following mosquito-sensitive-resistant model:

$$\dot{y} = cxq + c(1-u)xQ - cuyQ - ry, \tag{7.1}$$

$$\begin{aligned}
\dot{Y} &= cuxQ + cuyQ - RY, & (7.2)\\
\dot{z} &= RY - \gamma z, & (7.3)\\
\dot{q} &= \alpha z - \eta q, & (7.4)\\
\dot{Q} &= \beta z - \zeta Q. & (7.5)\\
\dot{x} &= -cx(q+Q) + ry + \gamma z, & (7.6)
\end{aligned}$$

where $x + y + Y + z = N$, γ is the rate of loss of immunity, c is the unit contact rate between susceptibles and infected mosquitoes with which new cases occur, δ is the death or removal rate of sensitive/ resistant type mosquitoes. Recovery from resistant infection with per capita rate R is simply a transition to the immune class. The contribution $cuyQ$ is a case of superinfection. Such phenomenon has been known to occur in reality (see for e.g. [12, 14]).

The principal evolutionary pressure favourable to the propagation of any of the strains is the geographical distribution of the region under consideration.

Eliminating x from (7.1) and (7.2) (using $x + y + Y + z = N$) we get:

$$\begin{aligned}
\dot{y} &= -ry + cNq + cN(1-u)Q - cyq - cYq - czq - c(1-u)yQ - c(1-u)YQ \\
&\quad -c(1-u)zQ - cuyQ, & (7.7)\\
\dot{Y} &= -RY + cuNQ - cu(Y+z)Q, & (7.8)
\end{aligned}$$

Equations (7.3), (7.4) (7.5), (7.7) and (7.8) have two equilibrium points, out of which the origin is one. The other equilibrium point is positive provided

$$N > max\left\{\frac{\gamma\zeta}{cu\beta}, \frac{[\alpha\zeta(\gamma+R) + \eta(1-u)\beta\gamma + \eta(1-u)\beta R]z_0}{\alpha\zeta R + \eta R(1-u)\beta}\right\}, \tag{7.9}$$

where,

$$z_0 = \frac{R(cu\beta N - \gamma\zeta)}{cu\beta(\gamma+R)}, \; Y_0 = \frac{\gamma}{R}z_0 = \frac{\gamma(cu\beta N - \gamma\zeta)}{cu\beta(\gamma+R)}. \tag{7.10}$$

Let $P_0 = (y_0, Y_0, z_0, q_0, Q_0)$ be the nontrivial equilibrium point of the system (7.3), (7.7), (7.8), (7.4) and (7.5). Then the Jacobian matrix of the system at P_0 is

$$J(P_0) = \begin{bmatrix} -(r+cq_0+cQ_0) & -c(q_0+(1-u)Q_0) & -c(q_0+(1-u)Q_0) & j_{14} & j_{15} \\ 0 & -(R+cuQ_0) & -cuQ_0 & 0 & -j_{25} \\ 0 & R & -\gamma & 0 & 0 \\ 0 & 0 & \alpha & -\eta & 0 \\ 0 & 0 & \beta & 0 & -\zeta \end{bmatrix}, \quad (7.11)$$

where,

$$j_{14} = c(N - y_0 - Y_0 - z_0),\ j_{15} = c[(1-u)(N - Y_0 - z_0) - y_0],\ j_{25} = cuN - cu(Y_0 + z_0)$$

and P_0 is given by

$$P_0 = (y_0, Y_0, z_0, q_0, Q_0) = (y_0, Y_0, \frac{R}{\gamma}Y_0, \frac{\alpha R}{\eta\gamma}Y_0, \frac{\beta R}{\zeta\gamma}Y_0), \quad (7.12)$$

with Y_0 as in equation (7.10).

The eigenvalues of the matrix $J(P_0)$ are

$$\lambda_1 = -(r + cq_0 + cQ_0) < 0,\ \lambda_2 = -\eta < 0,$$

and the three eigenvalues of the matrix

$$J_1 = \begin{bmatrix} -(R+cuQ_0) & -cuQ_0 & cuN - cu(Y_0+z_0) \\ R & -\gamma & 0 \\ 0 & \beta & -\zeta \end{bmatrix}. \quad (7.13)$$

The characteristic polynomial of J_1 is

$$\lambda^3 + (R + cuQ_0 + \gamma + \zeta)\lambda^2 + ([R+cuQ_0]\gamma + [R+cuQ_0]\zeta + \gamma\zeta + cuQ_0R)\lambda$$
$$+ (R+cuQ_0)\gamma\zeta + cuQ_0R\zeta + cu\beta R(Y_0 + z_0 - N).$$

The Routh–Hurwitz condition for all eigenvalues of J_1 to have negative real parts is

$$(R + cuQ_0 + \gamma + \zeta)(R\gamma + cuQ_0\gamma + R\zeta + cuQ_0\zeta + \gamma\zeta + cuQ_0R)$$
$$> cuQ_0R\zeta + (R + cuQ_0)\gamma\zeta + cu\beta R(Y_0 + z_0) - \beta RcuN > 0,$$

which simplifies to

$$N < \frac{Q_0\zeta}{\beta} + \frac{\gamma\zeta Q_0}{\beta R} + \frac{\gamma\zeta}{cu\beta} + Y_0 + z_0. \tag{7.14}$$

Since $\frac{Q_0\zeta}{\beta} + \frac{\gamma\zeta Q_0}{\beta R} + \frac{\gamma\zeta}{cu\beta} = N$, (7.14) further simplifies to

$$0 < Y_0 + z_0. \tag{7.15}$$

Thus the nontrivial equilibrium P_0 is asymptotically stable if and only if condition (7.15) holds.

By considering the linearized system near the origin it is easy to see that the origin is an asymptotic equilibrium point if and only if

$$N < \frac{\gamma\zeta}{cu\beta}. \tag{7.16}$$

It follows that the origin is an unstable equilibrium point if and only if

$N > \frac{\gamma\zeta}{cu\beta}$ or $cu\beta N > \gamma\zeta$. But if this is the case, equation (7.15) is satisfied since both Y_0 and z_0 automatically assume positive values. We state the ongoing discussion in a theorem as follows.

THEOREM 7.1

The origin is a locally asymptotically stable equilibrium point of the system (7.3), (7.7), (7.8), (7.4) and (7.5) provided that $N < \frac{\gamma\zeta}{cu\beta}$ when it is the only equilibrium point. If $N > \frac{\gamma\zeta}{cu\beta}$ then the origin is unstable and there is a unique endemic equilibrium $(y_0, Y_0, z_0, q_0, Q_0)$ (where $(y_0, Y_0, z_0, q_0, Q_0)$ is given by equations (7.10) and (7.12)) which is locally stable.

Hence the issue of the existence and stability of the unique nontrivial endemic state is not a global phenomenon; it is dependent on parameter variations. If there is no resistant infection (u=0) the only equilibrium point is the trivial one and it is asymptotically stable for all parameter values.

7.3 Multigroup population

In this section we consider a case of a k-interacting human population groups: $N_i, i = 1, 2, ..., k$, with $N = \sum_{i=1}^{k} N_i$. We assume that infecteds in each group receive initial treatment. Thus as in the 2-strain model, sensitive infecteds cannot produce gametocytes since the parasites are destroyed by drugs early enough before they get to the exoerythrocytic stage. The degree of resistivity of parasites to a particular antimalaria drug is a function of geographical distribution. Denote by y_i, Y_i, and z_i respectively the numbers of sensitive, resistant and (partial) immunes associated with group N_i; also denote by q_i and Q_i respectively sensitive and resistant vectors associated with group N_i. The emergence and propagation of resistant parasites in each group depend on certain evolutionary pressure. For simplicity we shall assume that the growth of each of the vector groups depends on the population of (partial) immunes in that group. Vectors from group j infect susceptibles of group i at a rate of c_{ij}. Let u_j be the probability that a susceptible in group i is infected by a resistant strain from group j; the complementary probability, $1 - u_j$, is the probability that a susceptible is infected with sensitive strain only. Let r_i and R_i respectively denote the cure rates for the sensitive and resistant infecteds. Note that resistant cure is just a transition to the corresponding immune class. The model for this k-population group is then described by the following system of differential equations:

$$\dot{y}_i = \sum_{i=1}^{k}[c_{ij}(N_i - y_i - Y_i - z_i)q_j + c_{ij}(1 - u_j)(N_i - y_i - Y_i - z_i)Q_j - c_{ij}u_j y_i Q_j] - r_i y_i \quad (7.17)$$

$$\dot{Y}_i = \sum_{i=1}^{k} u_j c_{ij} Q_j (N_i - Y_i - z_i) - R_i Y_i. \quad (7.18)$$

$$\dot{z}_i = R_i Y_i - \gamma_i z_i, \quad (7.19)$$

$$\dot{q}_i = \alpha_i z_i - \eta_i q_i, \quad (7.20)$$

$$\dot{Q}_i = \beta_i z_i - \zeta_i Q_i, \quad (7.21)$$

$c_{ij} \geq 0$, $0 < u_i \leq 1$, $r_i > 0$, $R_i > 0$, $\alpha_i > 0$, $\beta_i > 0$, $\gamma_i > 0$, $\eta_i > 0$, $\zeta_i > 0$ and $0 \leq y_i + Y_i + z_i \leq N_i$.

Let $\mathbf{X}=(y_1, ..., y_k, Y_1, ..., Y_k, z_1, ..., z_k, q_1, ..., q_k, Q_1, ..., Q_k)^T$. Then $\mathbf{X}$ is a 5k-dimensional vector whose first, second, third, fourth, and fifth k components are respectively those of y_i (i=1,...,k), Y_i

(i=1,...,k), z_i (i=1,...,k), q_i (i=1,...,k), and Q_i (i=1,...,k). The analysis of the system (7.17)—(7.21) is more complex than that of the corresponding 2-strain system. We begin with the following definitions/preliminaries.

For vectors a and b, where $a = (a_1, \ldots, a_k)^T$, $b = (b_1, \ldots, b_k)^T$, let $a \otimes b$ be the vector whose components are the products of the corresponding components of its arguments, i.e.

$$a \otimes b = (a_1 b_1, \ldots a_k b_k)^T.$$

It is easy to see that the operation $\otimes$ is symmetric, bilinear, associative and commutative. If diag(d) for a vector d means the diagonal matrix (of appropriate dimension) whose ith diagonal entry is d_i and 1 means the vector with all components equal to 1, then

$$a \otimes b = diag(a)b \quad \text{and} \quad a.b = (a \otimes b).1$$

(where . means dot product). If B is a matrix, $a \otimes B$ is a matrix with $(a \otimes B)_{ij} = a_i B_{ij}$.

In chapter 6 we extended Proposition PS to $\mathcal{R}^4$ and thus obtained a unique globally asymptotic equilibrium point for a system in $\mathcal{R}^4$.

Let $y = (y_1, \ldots, y_k)^T$, $Y = (Y_1, \ldots, Y_k)^T$, $z = (z_1, \ldots, z_k)^T$, $q = (q_1, \ldots, q_k)^T$, $Q = (Q_1, \ldots, Q_k)^T$. With these definitions, the vector form of system (7.17—(7.21) is

$$\begin{aligned}
\dot{y} &= -r \otimes y + (N \otimes C)q + (N \otimes C)((1-u) \otimes Q) - ((y + Y + z) \otimes C)q \\
&\quad -((y + Y + z) \otimes C)((1-u) \otimes Q) - (y \otimes C)(u \otimes Q), && (7.22)\\
\dot{Y} &= -R \otimes Y + (N \otimes C)(u \otimes Q) - ((Y + z) \otimes C)(u \otimes Q), && (7.23)\\
\dot{z} &= R \otimes Y - \gamma \otimes z, && (7.24)\\
\dot{q} &= \alpha \otimes z - \eta \otimes q, && (7.25)\\
\dot{Q} &= \beta \otimes z - \zeta \otimes Q. && (7.26)
\end{aligned}$$

The set of feasible values is

$$\Omega = \{\mathbf{X} = (y, Y, z, q, Q)^T \in \mathcal{R}^{5k} : y_i \geq 0, Y_i \geq 0, z_i \geq 0, q_i \geq 0, Q_i \geq 0, y_i + Y_i + z_i \leq N_i, \forall i\}.$$

THEOREM 7.2

The set Ω is positively invariant for the system (7.17)—(7.21).

Proof. By Nagumo's theorem [51], if the outward-pointing normal $\mathbf{n}$ at each point on the boundary $\partial\Omega$ of a set Ω when dot multiplied with the vector field $\dot{y}$ at that point yields a nonpositive result, then the set is positively invariant for the differential equation.

The boundary of Ω corresponding to the condition $y_i \geq 0$ or $-y_i \leq 0$ gives rise to the outward-pointing 5k-dimensional normal vector

$$\mathbf{n} = (0, ..., -1, ..., 0)^T, \qquad -1 \textit{ in the ith position.}$$

We are to show that

$$\mathbf{n}.\dot{\mathbf{X}} \leq 0$$

when $y_i = 0$. From equation (7.17), when $y_i = 0$, $-\dot{y}_i$ is

$$-(N_i - Y_i - z_i)\sum_{j=1}^{k} c_{ij}[q_j + (1 - u_j)Q_j] \leq 0.$$

The boundary of Ω corresponding to $-Y_i \leq 0$ gives rise to the 5k-dimensional normal vector

$$\mathbf{n} = (0, ..., -1, ..., 0)^T, \qquad -1 \textit{ in the } (k+i)\textit{th position.}$$

From equation (7.18), when $Y_i = 0$,

$$\mathbf{n}.\dot{\mathbf{X}} = -Y_i = -\sum_{j \neq i} u_j c_{ij} Q_j (N_i - z_i) \leq 0.$$

Similar results hold for the boundaries of Ω corresponding to

$$-z_i \leq 0, \quad -q_i \leq 0, \quad -Q_i \leq 0.$$

The boundary of Ω corresponding to $y_i + Y_i + z_i \leq N_i$ gives rise to the 5k-dimensional normal,

$$\mathbf{n} = (0, ..., 1, ..., 1, ..., 1, ...0)^T, \qquad 1's\ in\ the\ ith,\ (k+i)th\ and\ (2k+i)th\ positions.$$

At boundary, $y_i + Y_i + z_i = N_i$, so that $N_i - Y_i - z_i = y_i$; hence, from equations (7.17), (7.18) and (7.19)

$$\begin{aligned}
\mathbf{n}.\dot{\mathbf{X}} &= \dot{y}_i + \dot{Y}_i + \dot{z}_i \\
&= -\sum(c_{ij}u_j y_i Q_j - r_i y_i + \sum(c_{ij}u_j y_i Q_j) - R_i Y_i + R_i Y_i - \gamma_i z_i \\
&= -(r_i y_i + \gamma_i z_i) \leq 0.
\end{aligned}$$

Hence Ω is positively invariant for the system (7.17)—(7.21).

Now, back to system (7.17)—(7.21). We do not know a complete analytical solution to this problem. However, with much restrictions as below, we obtain some reasonable results.

At equilibrium, $\dot{z}_i = 0$, $\dot{q}_i = 0$, $\dot{Q}_i = 0$. This implies that

$$z_i = \frac{R_i}{\gamma_i} Y_i = \rho_i Y_i \qquad (\rho_i = \frac{R_i}{\gamma_i}), \tag{7.27}$$

$$q_i = \frac{\alpha_i}{\eta_i} z_i = \mu_i Y_i \qquad (\mu_i = \frac{\alpha_i R_i}{\eta_i \gamma_i}), \tag{7.28}$$

$$Q_i = \frac{\beta_i}{\zeta_i} z_i = \omega_i Y_i \qquad (\omega_i = \frac{\beta_i R_i}{\zeta_i \gamma_i}). \tag{7.29}$$

At this point, equation (7.18) becomes

$$\dot{Y}_i = \sum_{j=1}^{k} [c_{ij} u_j \omega_j Y_j (N_i - Y_i - \rho_i Y_i)] - R_i Y_i. \tag{7.30}$$

System (7.30) is amenable to the LY system with matrix of linear terms given by

$$A_Y = diag(N) C diag(u \otimes \omega) - diag(R), \tag{7.31}$$

and the vector of quadratic terms given by

$$T_Y = ((\rho + 1) \otimes Y \otimes C)(u \otimes \omega \otimes Y). \tag{7.32}$$

Thus Y tends asymptotically to 0 if $s(A_Y) \leq 0$ and to a fixed nonnegative vector Y^* (say) if $s(A_Y) > 0$.

Hence:

Q tends asymptotically to 0 if $s(A_Y) \leq 0$ and to $\omega \otimes Y^*$ if $s(A_Y) > 0$;

z tends asymptotically to 0 if $s(A_Y) \leq 0$ and to $\rho \otimes Y^*$ if $s(A_Y) > 0$;

q tends asymptotically to 0 if $s(A_Y) \leq 0$ and to $\mu \otimes Y^*$ if $s(A_Y) > 0$.

By substituting these values (in view of Proposition PS extended to $\mathcal{R}^{5k}$) into equation (7.22) we also get

y tends asymptotically to 0 if $s(A_Y) \leq 0$ and to $y^* \neq 0$ if $s(A_Y) > 0$.

We state this in a theorem as follows:

THEOREM 7.3

The solutions to system (7.17)—(7.21) approach the origin if $s(A_Y) \leq 0$ and approach a unique positive equilibrium $(y^*, Y^*, z^*, q^*, Q^*)^T$ provided $Y(0) \neq 0$; where $z^* = \rho \otimes Y^*$, $q^* = \mu \otimes Y^*$, $Q^* = \omega \otimes Y^*$ and y^* is obtained by setting the right hand side of equation (7.22) to zero and substituting for the known vectors.

It follows that both infections die out if $s(A_Y) \leq 0$ and that an endemic state is reached for both infections if $s(A_Y) > 0$. This result together with that of theorem 1 shows that both models have similar results. These results should be a guide to health officials. Since major health efforts should be geared towards eliminating the parasites (and hence the infection), conditions leading to the origin being an asymptotic equilibrium point should be put in place.

Finally we remark that a complete analytical solution of the system (7.17)—(7.21) is desirable.

Chapter 8

A Mathematical Model for Malaria Treating both Sensitive and Resistant Strains in a Spatially Distributed Population

8.1 Introduction

The emergence of drug-resistant malaria parasites in recent years has become a significant public health problem. Drawing from our ealier models (see chapters 6 and 7), a model for a spatially distributed population is hereby introduced. Our focus in analysing the models is on establishing some positive asymptotic equilibria. It is shown that (under suitable conditions) the equilibrium points are (globally) asymptotically stable. As a function of some interplay between the various parameters, the equilibrium can lead to endemic infection with sensitive infection only, resistant infection only, or both, or to elimination of both infections.

8.2 Spatially Distributed Population Model

Now, we consider a case of a heterogeneous interacting population groups: N_i, $i = 1, 2, ..., k$; $N = \sum_{i=1}^{k} N_i$. This consideration becomes necessary in view of the fact that resistivity of parasites to a particular antimalaria drug is a function of geographical distribution. We denote by y_i and Y_i respectively sensitive and resistant infecteds associated with group N_i; the corresponding vectors are denoted by q_i and Q_i respectively. Vectors from group j infect susceptibles of group i at a rate of c_{ij}. The probability that a susceptible in group i is infected by a resistant vector from group j is denoted by u_j; the complementary probability is $1 - u_j$. The model for this spatial heterogeneity is then described by the following system of differential equations:

$$\begin{aligned}
\dot{y}_i &= \sum_{j=1}^{k}[c_{ij}N_i q_j + c_{ij}N_i(1-u_j)Q_j - c_{ij}q_j(y_i+Y_i) - c_{ij}y_iQ_j - c_{ij}(1-u_j)Y_iQ_j] - r_iy_i & (8.1)\\
\dot{Y}_i &= \sum_{j=1}^{k}[u_jc_{ij}Q_j(N_i - Y_i)] - R_iY_i, & (8.2)\\
\dot{q}_i &= v_i - \delta_i q_i, & (8.3)\\
\dot{Q}_i &= V_i - \Delta_i Q_i. & (8.4)
\end{aligned}$$

If vectors grow by factors proportional to the number of infecteds then equations (8.3) and (8.4) are replaced by

$$\begin{aligned}
\dot{q}_i &= \alpha_{1i}y_i + \alpha_{2i}Y_i - \delta_i q_i, & (8.5)\\
\dot{Q}_i &= \gamma_i Y_i - \Delta_i Q_i,; & (8.6)
\end{aligned}$$

$c_{ij} \geq 0$, $0 < u_i \leq 1$, $r_i > 0$, $R_i > 0$, $\alpha_{1i}, \alpha_{2i} > 0$, $\delta_i > 0$, $\Delta_i > 0$, $\gamma_i > 0$ and $0 \leq y_i + Y_i \leq N_i$.

Let $\mathbf{X} = (y_1, ..., y_k, Y_1, ..., Y_k, q_1, ..., q_k, Q_1, ..., Q_k)^T$.
Then $\mathbf{X}$ is a 4k-dimensional vector whose first, second, third, and fourth k components are respectively those of $y_i (i = 1, ..., k)$, $Y_i (i = 1, ..., k)$, $q_i (i = 1, ..., k)$, and $Q_i (i = 1, ..., k)$.

For the stability analysis of the system (8.1)–(8.4) we shall generalize Proposition PS so that $(x_1, x_2) \in \mathcal{R}^{4k}$ and $x_2 \in \mathcal{R}^{3k}$. As can be seen from the system, both vector groups are each disconnected from the system as a whole.

Clearly $q_i \to \frac{v_i}{\delta_i}$, $\quad Q_i \to \frac{V_i}{\Delta_i}$ as $t \to \infty$. By application of Proposition PS with $x_2 \in \mathcal{R}^{2k}$ to the equations (8.2) and (8.4) we deduce

$Y_i \to \frac{N_i \sum_{j=1}^{k} u_j c_{ij} \frac{V_j}{\Delta_j}}{\sum_{j=1}^{k} u_j c_{ij} \frac{V_j}{\Delta_j} + R_i}$. Hence equation (8.1) has the form

$$\begin{aligned} \dot{y}_i &= -(r_i + \sum_{j=1}^{k}[c_{ij}q_j + c_{ij}Q_j])y_i + (N_i - Y_i)\sum_{j=1}^{k} c_{ij}[q_j + (1-u_j)Q_j] \\ \text{or} \quad \dot{y}_i &= -A_i(t)y_i + B_i(t) \end{aligned}$$

where

$$A_i(t) \to A_i = r_i + \sum_{j=1}^{k}[c_{ij}\frac{v_j}{\delta_j} + c_{ij}\frac{V_j}{\Delta_j}] > 0 \text{ as } t \to \infty$$

and

$$B_i(t) \to B_i = (N_i - Y_i)\sum_{j=1}^{k} c_{ij}[\frac{v_j}{\delta_j} + (1-u_j)\frac{V_j}{\Delta_j}] > 0 \quad \text{as} \quad t \to \infty.$$

The rest of the proof follows by mimicking the idea used by the author in a previous chapter 6, section 3.

Eventually we arrive at $y_i \to \frac{B_i}{A_i}$ as $t \to \infty$ which is the required result. Hence the system (8.1)–(8.4) has a unique positive globally asymptotic equilibrium.

For the analysis of the system (8.1), (8.2), (8.5), and (8.6), we begin with the following definitions:

For vectors **a** and **b**, where $\mathbf{a} = (a_1, ..., a_k)^T$, $\mathbf{b} = (b_1, ..., b_k)^T$, let $\mathbf{a} \otimes \mathbf{b}$ be the vector whose components are the products of the corresponding components of its arguments, i.e.

$$\mathbf{a} \otimes \mathbf{b} = (a_1 b_1, \ ..., \ a_k b_k)^T.$$

It is easy to see that the operation $\otimes$ is symmetric, bilinear, associative and commutative. If diag(**d**) for a vector **d** means the diagonal matrix (of appropriate dimension) whose ith diagonal entry is d_i and $\mathbf{1}$ means the vector with all components equal to 1, then

$\mathbf{a} \otimes \mathbf{b}$=diag($\mathbf{a}$)$\mathbf{b}$ and

$\mathbf{a}.\mathbf{b} = (\mathbf{a} \otimes \mathbf{b}).\mathbf{1}$ (where . means dot product). If $\mathbf{B}$ is a matrix, $\mathbf{a} \otimes \mathbf{B}$ is a matrix with $(\mathbf{a} \otimes \mathbf{B})_{ij} = a_i B_{ij}$. Let $\mathbf{y} = (y_1, ..., y_k)^T$, $\mathbf{Y} = (Y_1, ..., Y_k)^T$, $\mathbf{q} = (q_1, ..., q_k)^T$, $\mathbf{Q} = (Q_1, ..., Q_k)^T$. With these definitions, the vector form of system (8.1), (8.2), (8.5), and (8.6) is:

$$\begin{aligned}
\dot{\mathbf{y}} &= -\mathbf{r} \otimes \mathbf{y} + (\mathbf{N} \otimes \mathbf{C})\mathbf{q} + (\mathbf{N} \otimes \mathbf{C})((\mathbf{1} - \mathbf{u}) \otimes \mathbf{Q}) - ((\mathbf{y} + \mathbf{Y}) \otimes \mathbf{C})\mathbf{q} \\
&\quad -(\mathbf{y} \otimes \mathbf{C})\mathbf{Q} - (\mathbf{Y} \otimes \mathbf{C})((\mathbf{1} - \mathbf{u}) \otimes \mathbf{Q}), \qquad (8.7) \\
\dot{\mathbf{Y}} &= -\mathbf{R} \otimes \mathbf{Y} + (\mathbf{N} \otimes \mathbf{C})(\mathbf{u} \otimes \mathbf{Q}) - (\mathbf{Y} \otimes \mathbf{C})(\mathbf{u} \otimes \mathbf{Q}), \qquad (8.8) \\
\dot{\mathbf{q}} &= \alpha_1 \otimes \mathbf{y} + \alpha_2 \otimes \mathbf{Y} - \delta \otimes \mathbf{q}, \qquad (8.9) \\
\dot{\mathbf{Q}} &= \gamma \otimes \mathbf{Y} - \mathbf{\Delta} \otimes \mathbf{Q}, \qquad (8.10)
\end{aligned}$$

Now, back to system (8.1), (8.2), (8.5), and (8.6). I do not know a complete analytical characterization of this system. However, with much restrictions as below, we arrive at some reasonable results:

If Q_i is fixed at its equilibrium, i.e. $\gamma_i Y_i - \Delta_i Q_i = 0$

$$\Rightarrow Q_i = \frac{\gamma_i}{\Delta_i} Y_i = \eta_i Y_i \ (where \ \eta_i = \frac{\gamma_i}{\Delta_i}). \qquad (8.11)$$

At this point, equation (8.2) becomes identical to equation (1.7). It follows that the resistant infecteds' subsystem is equivalent to the LY system (equation (1.7), see our General Introduction); hence the resistant infecteds' subsystem tends asymptotically to a fixed nonnegative vector, say $\mathbf{Y}^*$. From (8.11), this also means that the resistant vectors' subsystem tends asymptotically to a fixed vector, $\eta \otimes \mathbf{Y}^*$, where $\eta_i = \frac{\gamma_i}{\Delta_i}$. From equation (8.2), the matrix of coefficients for the resistant infecteds, which is disconnected from that of the sensitive infecteds, is

$$\mathbf{A}_Y = diag(\mathbf{N})\mathbf{C}diag(\mathbf{u} \otimes \eta) - diag(\mathbf{R})$$

and the vector of quadratic terms is

$$\mathbf{T} = (\mathbf{Y} \otimes \mathbf{C})(\mathbf{u} \otimes \eta \otimes \mathbf{Y}).$$

Thus, from theorem LY,

$\mathbf{Y}^* = \mathbf{0}$ if $s(\mathbf{A}_Y) \leq 0$, and $\mathbf{Y}^* \neq 0$ if $s(\mathbf{A}_Y) > 0$.

Hence, $\mathbf{Q}^* = \mathbf{0}$ if $s(\mathbf{A}_Y) \leq 0$ and $\mathbf{Q}^* \neq 0$ if $s(\mathbf{A}_Y) > 0$.

Again, note that if $(\mathbf{Y}(\mathbf{0}), \mathbf{Q}(\mathbf{0}))^T = (\mathbf{0}, \mathbf{0})^T$, then $\dot{\mathbf{q}} = \mathbf{0} \Rightarrow q_i = \frac{\alpha_i}{\delta_i} y_i = \xi_i y_i$. At this point, $\dot{y}_i = \left(\sum_{j=1}^{k} c_{ij}\xi_j y_j (N_i - y_i)\right) - r_i y_i$.

In this case, the sensitive infected humans' subsystem becomes identical to system (1.7) and hence it tends asymptotically to a fixed nonnegative vector $\mathbf{y}^*$. The matrix of coefficients for the linear terms of the sensitive vectors when $\mathbf{Y}^T = \mathbf{0}^T$, $\mathbf{Q}^T = \mathbf{0}^T$, $\mathbf{q}^T = \mathbf{0}^T$ is

$$\mathbf{A}_y = diag(\mathbf{N})\mathbf{C}diag(\xi) - diag(\mathbf{r}).$$

and the vector of quadratic terms is

$$\mathbf{W}_y = (\mathbf{y} \otimes \mathbf{C})(\xi \otimes \mathbf{y}).$$

$\mathbf{y}$ tends asymptotically to $\mathbf{0}$ if $s(\mathbf{A}_y) \leq 0$ and to a fixed nonnegative vector $\mathbf{y}^*$ if $s(\mathbf{A}_y) > 0$. Hence, both infections die out if both $s(\mathbf{A}_y) \leq 0$ and $s(\mathbf{A}_Y) \leq 0$. Thus, our subconclusion is as follows:

THEOREM 8.1

(i) $(\mathbf{y}, \mathbf{Y}, \mathbf{q}, \mathbf{Q})^T = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})^T$ is asymptotically stable if $s(\mathbf{A}_Y) \leq 0$ and $s(\mathbf{A}_y) \leq 0$.

(ii) $(\mathbf{y}, \mathbf{Y}, \mathbf{q}, \mathbf{Q})^T = (\mathbf{y}^*, \mathbf{0}, \mathbf{q}^*, \mathbf{0})^T$, (where $\mathbf{y}^*$, $\mathbf{q}^*$ are some positive vectors) is asymptotically stable if $s(\mathbf{A}_Y) \leq 0$ and $s(\mathbf{A}_y) > 0$.

Next, we examine the case, $s(\mathbf{A}_Y) > 0$. Let $\mathbf{E}_1 = \mathbf{Y} - \mathbf{Y}^*$, $\mathbf{E}_2 = \mathbf{Q} - \mathbf{Q}^*$. Then the equation for the sensitive (human and vectors) subsystems become

$$\begin{aligned}\dot{\mathbf{y}} &= -\mathbf{r}\otimes\mathbf{y}+(\mathbf{N}\otimes\mathbf{C})\mathbf{q}+(\mathbf{N}\otimes\mathbf{C})\left((\mathbf{1}-\mathbf{u})\otimes(\mathbf{Q}^*+\mathbf{E_2})\right)-\left((\mathbf{y}+\mathbf{Y}^*+\mathbf{E}_1)\otimes\mathbf{C}\right)\mathbf{q}\\ &\quad-(\mathbf{y}\otimes\mathbf{C})(\mathbf{Q}^*+\mathbf{E_2})-\left((\mathbf{Y}^*+\mathbf{E}_1)\otimes\mathbf{C}\right)\left((\mathbf{1}-\mathbf{u})\otimes(\mathbf{Q}^*+\mathbf{E_2})\right), \qquad (8.12)\\ \dot{\mathbf{q}} &= \alpha_1\otimes\mathbf{y}+\alpha_2\otimes(\mathbf{Y}^*+\mathbf{E}_1)-\delta\otimes\mathbf{q}. \qquad (8.13)\end{aligned}$$

At equilibrium, $\dot{y}=0$, $\dot{q}=0$, E_1, $E_2\to 0$. From (8.12) and (8.13), this condition leads to a vector quadratic equation in each of $\mathbf{y}$ and $\mathbf{q}$, whose complete analytical characterization is elusive. However, for special cases such as $\mathbf{u}=\mathbf{1}$, $\alpha_2=\mathbf{0}$, we again obtain asymptotic stability as follows: If $\mathbf{u}=\mathbf{1}$, $\alpha_2=0$, then at equilibrium $q_i=\frac{\alpha_{1i}y_i}{\delta_i}=\omega_i y_i$ (say). At this point, we see from equation (8.12) that

$$\dot{\mathbf{y}}=-\mathbf{r}\otimes\mathbf{y}+(\mathbf{N}\otimes\mathbf{C})(\omega\otimes\mathbf{y})-\left((\mathbf{y}+\mathbf{Y}^*+\mathbf{E}_1)\otimes\mathbf{C}\right)(\omega\otimes\mathbf{y})-(\mathbf{y}\otimes\mathbf{C})(\mathbf{Q}^*+\mathbf{E_2}).$$

Now, $o(E_1,E_2)\to 0$ as $\|E_1\|$, $\|E_2\|\to 0$. Hence

$$\dot{\mathbf{y}}=((\mathbf{N}-\mathbf{y})\otimes\mathbf{C})(\omega\otimes\mathbf{y})-(\mathbf{C}\mathbf{Q}^*+\mathbf{r})\otimes\mathbf{y}-(\mathbf{Y}^*\otimes\mathbf{C})(\omega\otimes\mathbf{y}). \qquad (8.14)$$

Equation (8.14) can be identified with system (??) with the matrix

$$\mathbf{B}_y=diag(\mathbf{N}-\mathbf{Y}^*)\mathbf{C}diag(\omega)-diag(\mathbf{C}\mathbf{Q}^*+\mathbf{r}), \qquad (8.15)$$

as the matrix of coefficients for the linear term and $\mathbf{Z}_y=-(\mathbf{y}\otimes\mathbf{C})(\omega\otimes\mathbf{y})$, as the vector of quadratic terms.

Thus, $\mathbf{y}\to\mathbf{0}$ and consequently, $\mathbf{q}\to\mathbf{0}$ if $s(\mathbf{B})\le 0$

and

$\mathbf{y}\to\mathbf{y}^*\neq\mathbf{0}$ and consequently $\mathbf{q}\to\omega\otimes\mathbf{y}^*$ if $s(\mathbf{B}_y)\ge 0$.

Hence, we adjoin to the subconclusions in theorem 1 the following similar results and thus obtain a more general conclusion as regards the stability analysis of the system (8.1), (8.2), (8.5) and (8.6):

THEOREM 8.2

(i) $(\mathbf{y}, \mathbf{Y}, \mathbf{q}, \mathbf{Q})^T = (\mathbf{0}, \mathbf{Y}^*, \mathbf{0}, \mathbf{Q}^*)^T$ is asymptotically stable if $s(\mathbf{A}_Y) > 0$ and **u=1**, $\alpha_2 = 0$ leads to $s(\mathbf{B}_y) \leq 0$.

(ii) $(\mathbf{y}, \mathbf{Y}, \mathbf{q}, \mathbf{Q})^T = (\mathbf{y}^*, \mathbf{Y}^*, \mathbf{q}^*, \mathbf{Q}^*)^T$ is asymptotically stable if $s(\mathbf{A}_Y) > 0$ and **u=1**, $\alpha_2 = 0$ leads to $s(\mathbf{B}) > 0$, where $\mathbf{B}_y$ is the matrix in (8.15).

Thus, in case (i), sensitive infection dies out and in case (ii), an endemic state is reached for both infections.

Remark 8.1 A complete analytical characterization/solution of the system (8.1), (8.2), (8.5) and (8.6) proves elusive. It is certainly of interest to know the asymptotic limit(s) for all possible **u** when $s(\mathbf{A}_Y) > 0$, and $s(\mathbf{B}_y) > 0$.

8.3 Conclusion

This chapter focuses on malaria infection in a multigroup population. The first model is typical of a situation in which there is a sudden out break of infection for each of the population groups. In this case, infected vectors arrive at a constant rate from the rest of the groups. This model shows that resistant infectives persist and nearly everybody could be infected with the resistant parasite if the death rate of resistant vectors is small. This is seen from the fact that $Y_i(t) \to N_i$ as $\Delta_i \to 0$. Since resistant infectives are less sensitive to the drug, major health efforts should be geared towards destroying the parasites.

The second multigroup model describes more accurately the current trend in malaria infection, especially in Sub-Saharan Africa, for instance, where strains that are resistant to virtually all known drugs have emerged. Since efforts should be directed towards eliminating resistant parasites (and hence resistant infection), conditions leading to the results of theorem 1 should serve as guides to a health management board. Condition (i) of the theorem shows that eventually both strains of the parasites are eliminated. Condition (ii) shows that resistant parasites die off. On the other hands, theorem 2 shows that resistant parasites/infection persist in both conditions of the theorem. Condition (ii) shows that an endemic state is reached for both infections.

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