CONVERGENCE IN NORM OF MODIFIED KRASNOSELSKII-MANN ITERATION FOR FIXED POINTS OF ASYMPTOTICALLY DEMICONTRACTIVE MAPPINGS

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Certification

This is to certify that ODIBO SAMUEL, a postgraduate student in the Department of Mathematics with registration number PG/M.Sc./10/52493 has satisfactorily completed the requirements for research work for the award of Master of Science Degree in Mathematics. The work embodied in this project report is original and has not been submitted in part or full for any other diploma or degree of this or any other university.

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Dedication

This work is dedicated to God Almighty who gave me life and health during the lecture periods as well as the inspiration l received while l was writing this project report.

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Contents

Abstract

This project report deals with the class of asymptotically demicontractive mappings in Hilbert spaces. We noted some historical aspects concerning the concept of asymptotically demicontractivity and studied a regularized variant of the Krasnoselskii-Mann iteration scheme, which ensured the strong convergence of the generated sequence towards the least norm element of the set of fixed points of asymptotically demicontractive mapping.

Chapter 1

Introduction

1.1 General Introduction

For the past 30 years or so, the study of Krasnoselskii-Mann iterative procedure for the approximation of fixed points of nonexpansive mappings and fixed points of some generalisations of nonexpansive mappings have been flourishing areas of research for many Mathematicians. Recently, Mainge and Maruster [22], studied the convergence in norm of modified Krasnoselkii-Mann iterations for fixed points of demicontractive mapping. In summary, they considered the following algorithm in a real Hilbert space H:

$$x_1 \in H,\ x_{n+1} = (1-\alpha_n)(1-\beta_n)x_n + \alpha_n T[(1-\beta_n)x_n] \quad n \geq 1 \tag{1.1.1}$$

The above equation can also be expressed in the form:

$$\begin{aligned} x_1 \in H,\ v_n &= (1-\beta_n)x_n \\ x_{n+1} &= (1-\alpha_n)v_n + \alpha_n T v_n \quad n \geq 1 \end{aligned} \tag{1.1.2}$$

They proved that if $T : H \to H$ is demicontractive, that is, $||Tx-p||^2 \leq ||x-p||^2 + k||x - Tx||^2$ for some $k \in [0,1)$, $\forall\ x \in H, \forall\ p \in F(T)$, where $F(T)$ (assumed to be nonempty) is the set of fixed points of T, then $\{x_n\}$ converges strongly to the least norm element of $F(T)$ under some mild assumptions on the iterative parameters $\{\alpha_n\}, \{\beta_n\}$ which are real sequences in $(0,1)$. We observe that the algorithm (1.1.2) will reduce to Mann iteration if

$\beta_n = 0$. That is, if $\beta_n = 0, \forall\, n \geq 1$, (1.1.2) reduces to

$$x_1 \in H,\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n \quad n \geq 1 \tag{1.1.3}$$

The important thing here is that their algorithm yielded strong convergence, but the Mann iterative algorithm yields only weak convergence. It was shown, however, that for the Mann iterative algorithm,$\|x_n - Tx_n\| \to 0 \;\; as\; n \to \infty$, that is, the sequence $\{x_n\}_{n\geq 1}$ generated by Mann's algorithm is an approximate fixed point sequence of nonexpansive operator T. Thus, to obtain strong convergence with Mann algorithm, additional requirements on T and/or its domain are usually required.

1.2 Aim and Scope of Study

The aim of this project is to propose an algorithm (analogous to (1.1.2)) above for asymptotically demicontractive mappings which will yield strong convergence; and in particular show that the proposed algorithm converges strongly to the least norm element of the set of fixed points of asymptotically demicontractive mappings.

Definition 1.2.1 *Let X be a nonempty set and let $T : X \to X$ be a self map. A fixed point of T is an element $x \in X$ for which $Tx = x$. The set of fixed points of T is denoted by $F(T)$. That is, $F(T) = \{x \in X : Tx = x\}$.*

Example 1.2.2 *Let $T : \mathbb{R} \to \mathbb{R}$ be a real valued function defined by $T(x) = x^2$, then $F(T) = \{x \in X : Tx = x\} = \{0, 1\}$*

Remark 1.2.3 *We quickly remark that not all functions have fixed points. For example, Let $f : \mathbb{R} \to \mathbb{R}$, be a real valued function defined by $f(x) = x + 1$, then f has no fixed point since there is no $x^* \in \mathbb{R}$ such that $f(x^*) = x^*$, that is, there is no $x^* \in \mathbb{R}$ such that $x^* + 1 = x^*$.*

Definition 1.2.4 *A vector space, or linear space, over a scalar field $F = \mathbb{R}$ or $\mathbb{C}$ is a nonempty set X, whose elements are called vectors, together with two operations, addition and multiplication by scalars,*

$$+ : X \times X \to X \quad \text{and} \quad . : \mathbb{R} \times X \to X$$

defined by $\qquad (x, y) \to x + y \quad \text{and} \quad (t, x) \to t.x,$

respectively, with the properties that:

1. *(X, +) is a commutative group, that is,*

   (a) $x + y = y + x \quad \forall\, \mathrm{x}, \mathrm{y} \in X$ *(commutative property)*

   (b) $x + (y + z) = (x + y) + z \quad \forall\, \mathrm{x}, \mathrm{y}, \mathrm{z} \in X$ *(associative property)*

   (c) *there is a vector* $0 \in X$ *called zero element, such that* $x + 0 = 0 + x \quad \forall\, x \in X$

   (d) *for every* $x \in X$ *there exists a vector* $-x \in X$*, called the additive inverse of* x *such that* $x + (-x) = (-x) + x = 0.$

2. $\forall\, x, y \in X$ *and* $\alpha,\ \beta \in \mathbb{R}$,

   (a) $\alpha.(x + y)) = \alpha.x + \alpha.y$

   (b) $(\alpha + \beta).x = \alpha.x + \beta.y$

   (c) $(\alpha.\beta).x = \alpha.(\beta.x)$

   (d) $1.x = x$ *for* $1 \in \mathbb{R}, x \in X$

Definition 1.2.5 *A normed space is a pair* $(X, ||.||)$*, where* X *is a vector space over the scalar field* $F = (\mathbb{R}$ *or* $\mathbb{C})$ *and* $||.|| : X \to \mathbb{R}$ *is a function satisfying the following conditions:*

1. $\|x\| \geq 0 \quad \forall\, x \in X,$

2. $\|x\| = 0$ *if and only if* $x = 0$ *in* X,

3. $\|\alpha x\| = |\alpha| \|x\| \qquad \forall\, \mathrm{x} \in X,\ \forall\, \alpha \in \mathbb{R},$

4. $\|x+y\| \leq \|x\| + \|y\| \; \forall \, \mathrm{x, y} \in X$.

Remark 1.2.6 *In normed spaces, the norm defines a metric, that is, if* $\|.\| : X \times X \to [0, \infty)$ *is a norm on* X *then for all* $x, y \in X$, *the function* $\|.\|$ *induces a metric given by*

$$\rho(x, y) = \|x - y\|.$$

Thus, we have a notion of completeness in terms of the norm. We say that $(X, \|.\|)$ is a Banach space if and only if $\rho(x, y)$ is a complete metric space, that is, if every Cauchy sequence $\{x_n\}$ in (X, ρ) converges to some element, say $x \in X$.

Definition 1.2.7 *Let* X *be a vector space over the field of scalars* $F = (\mathbb{R}$ *or* $\mathbb{C})$. *An inner product* $\langle ., . \rangle : X \times X \to F$ *on* X *is a function with domain* $X \times X$ *and range* F *such that the following conditions are satisfied:*

1. $\langle x, x \rangle \geq 0 \; \forall \, x \in X$,
2. $\langle x, x \rangle \; = 0$ *if and only if* $x = 0$,
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$,
4. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle \quad \forall \, x, y, z \in X$,
5. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \quad \forall \, x, y \in X, \quad \forall \, \alpha \in F$.

The pair $(X, \langle ., . \rangle)$ *is called an inner product space.*

Remark 1.2.8 *In number* 3 *of Definition 1.2.7, bar denotes complex conjugate. Thus, if* X *is a real Vector space, then* $\langle x, y \rangle = \langle y, x \rangle$.

Lemma 1.1

Let $(X, \langle ., . \rangle)$ be an inner product space. The function $\|.\| : X \to [0, \infty)$ defined for each $x \in X$ by

$$||x|| = \langle x, x \rangle^{\frac{1}{2}} \tag{1.2.1}$$

is a norm on X. Thus, we have a notion of completeness in terms of the norm. We say that X is a Hilbert space if it is a complete inner product space, that is, if every Cauchy sequence $\{x_n\} \subset X$ converges to some element, say $x \in X$.

Example 1.2.9 *Let $X = \mathbb{R}$, the set of real numbers with the usual addition and scalar multiplication. Define $\langle .,. \rangle : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \forall\, x, y \in \mathbb{R}$ by $\langle x, y \rangle = xy$, then $\langle .,. \rangle$ is an inner product on $\mathbb{R}$ and $(\mathbb{R}, \langle .,. \rangle)$ is a Hilbert space.*

Example 1.2.10 *Let $X := l_2(\mathbb{C}) = \{z = \{z_j\}_{j=1}^{\infty} : z_j \in \mathbb{C},\ j = 1, 2, 3, ...\},\ \sum_{j=1}^{\infty} |z_j|^2 < \infty\}$. Define $\langle .,. \rangle : X \times X \to \mathbb{C} \quad \forall\, z, w \in X$ by $\langle z, w \rangle = \sum_{j=1}^{\infty} z_j \overline{w}_j$, then $\langle .,. \rangle$ is an inner product on X and $(X, \langle .,. \rangle)$ is a Hilbert space*

We remark that not all inner product spaces are Hilbert spaces. For example, take $X = C[a, b] = \{f : [a, b] \to \mathbb{R} | f \textit{ is continuous}\}$ with the usual pointwise addition and scalar multiplication,

Define $\langle .,. \rangle : X \times X \to \mathbb{R} \quad \forall\, f, g \in X$ *by*

$$\langle f, g \rangle = \int_a^b f(t)g(t)dt$$

then $\langle .,. \rangle$ is an inner product on X so that the pair $(X, \langle .,. \rangle)$ is an inner product space that is not a Hilbert space.

Definition 1.2.11 *Let X be a vector space over the real numbers. A subset $K \subseteq X$ is said to be convex if for all $x, y \in K$ and all t in the interval $[0, 1]$, the point $(1 - t)x + ty \in K$. In other words, every point on the line segment connecting x and y is in K.*

Definition 1.2.12 *A mapping $T : C \to C$ is called demiclosed at zero if for any sequence $\{x_n\}_{n=1}^{\infty} \subset C$ which converges weakly to an $x \in C$, where C is closed and convex, the strong convergence of the sequence $\{Tx_n\}_{n=1}^{\infty}$ to zero in C implies that $Tx = 0$*

Definition 1.2.13 *Let X and Y be Banach spaces. A mapping $T : X \to Y$ is called completely continuous, if it maps a weakly convergent sequence in X to a strongly convergent sequence in Y. That is, $x_n \rightharpoonup x$* implies $\|Tx_n - Tx\|_Y \to 0$, as n $\to \infty$.

Proposition 1.2.14 *Let X and Y be normed linear spaces and $B(X,Y)$ denote the collection of all linear and bounded mappings from X to Y, that is, $B(X,Y) = \{T : X \to Y | T$ is linear and bounded$\}$. Let $B(X,Y)$ be endowed with the following norm, $\|T\|_{B(X,Y)} = sup\{\|T(x)\|_Y : \|x\|_X \leq 1\}$. Then $B(X,Y)$ is a Banach space if Y is.*

Proof:

Let $\{T_n\}_{n=1}^{\infty}$ be an arbitrary Cauchy sequence in $B(X,Y)$, i.e., $||T_n - T_m|| \to 0$ *as* $n,\ m \to \infty$. For each $x \in X$,

$$\begin{aligned} ||T_n(x) - T_m(x)||_Y &= ||(T_n - T_m)(x)||_Y \qquad \text{(definition of subtraction in B(X, Y))} \\ &\leq ||T_n - T_m||_{B(X,Y)}||x||_X \qquad \text{(T is bounded)} \\ &\to 0 \quad as\ n,\ m \to \infty \end{aligned}$$

Thus, for each $x \in X$, $\{T_n\}_{n=1}^{\infty}$ is a cauchy sequence in Y. Hence, for each $x \in X$, there exists $z_x \in Y$ such that $\{T_{n(x)}\}_{n=1}^{\infty}$ converges to $z_x \in Y$. Now, for each $x \in X$, set $Tx = z_x$. Then for any $x,\ y \in X$ and α a scalar, we have that

$$\begin{aligned} T(\alpha x + y) &= \lim_{n\to\infty} T_n(\alpha x + y) \\ &= \lim_{n\to\infty} (\alpha T_n(x) + T_n(y)) \quad \text{(since } \mathrm{T_n} \text{ is linear } \forall \text{ n} \in \mathbb{N}) \\ &= \alpha \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} T_n(y) \quad \text{(linearity of limit)} \\ &= \alpha Tx + Ty \end{aligned}$$

It follows that $T : X \to Y$ is a linear map. Next we show that T is bounded. To see this, observe that since $\{T_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $B(X,Y)$, then $\forall\ \epsilon > 0,\ \exists\ N_\epsilon \in \mathbb{N}\ \ni$ $\forall\ m, n \geq\ N_\epsilon$,

$$\|T_n - T_m\|_{B(X,Y)} < \epsilon.$$

In particular, for $\epsilon = 1,\ \exists\ N_1 \in \mathbb{N}$ such that $\forall\ m, n \geq N_1$,

$$\|T_n - T_m\|_{B(X,Y)} < 1.$$

This implies that $\forall\, n > N_1$,

$$\|T_n - T_{N_1}\|_{B(X,Y)} < 1.$$

Thus, $\forall\, n \geq N_1$, and for each $x \in X$, we obtain that

$$\begin{aligned}
\|T_n(x)\|_Y &\leq \|T_n(x) - T_{N_1}(x)\|_Y + \|T_n(x)\|_Y \\
&= \|(T_n - T_{N_1})(x)\|_Y + \|T_{N_1}(x)\|_Y \\
&\leq \|T_n - T_{N_1}\|_{B(X,Y)}\|x\|_X + \|T_{N_1}\|_{B(X,Y)}\|x\|_X \\
&< \|x\|_X + \|T_{N_1}\|_{B(X,Y)}\|x\|_X \\
&= (1 + \|T_{N_1}\|_{B(X,Y)})\|x\|_X \qquad (i)
\end{aligned}$$

Thus, we obtain from (i) that

$$\lim_{n\to\infty} \|T_n(x)\|_Y \leq (1 + \|T_{N_1}\|_{B(X,Y)})\|x\|_X$$

This implies (using continuity of norm) that

$$\|T(x)\|_Y \leq (1 + \|T_{N_1}\|_{B(X,Y)})\|x\|_X$$

Thus, $T \in B(X, Y)$.

Next, we show that indeed, $\{T_n\}_{n=1}^{\infty}$ converges to T with respect to the norm in $B(X, Y)$.
Let $\epsilon > 0$ be given. Then $\exists\, N_\epsilon \in \mathbb{N}$ such that $\forall\, m, n \geq N_\epsilon$,

$$\|T_n - T_m\|_{B(X,Y)} < \epsilon.$$

By continuity of the norm and for each $x \in X$ and $n > N_\epsilon$, we obtain

$$\begin{aligned}
\|T_n(x) - T(x)\|_{B(X,Y)} &= \lim_{m\to\infty} \|T_n(x) - T_m(x)\|_{B(X,Y)} \\
&\leq \lim_{m\to\infty} \|T_n - T_m\|_{B(X,Y)}\|x\|_X \\
&< \epsilon\|x\|_X
\end{aligned}$$

Taking supremum over $x \in X$ with $\|x\| \leq 1$ yields

$$\|T_n - T\| \leq \epsilon \quad \text{for all n > N}_\epsilon$$

Thus, $\{T_n\}_{n=1}^{\infty}$ converges to T in $B(X, Y)$.

1.3 Some Useful Identities and Inequalities in Hilbert Spaces

The following inequalities and identities are inequalities and identities in the literature.

Cauchy Schwartz Inequality (see e.g:[39])

Let $(X, \langle ., . \rangle)$ be an inner product space. Then for all $x, y \in X$, we have

$$| \langle x, y \rangle | \leq \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} \tag{1.3.1}$$

Parallelogram Identity (see e.g: [18])

Let $(X, \langle ., . \rangle)$ be an inner product space. Then for all $x, y \in X$, we have

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2) \tag{1.3.2}$$

Convex Identity(see e.g: [31])

Let $(X, \langle ., . \rangle)$ be an inner product space over the scalar field of $\mathbb{R}$. Then for all $x, y \in X$ and $\lambda \in [0, 1]$ we have

$$||\lambda x + (1 - \lambda)y||^2 = \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)||x - y||^2 \tag{1.3.3}$$

Remark 1.3.1 *It can be easily verified that if we set $\lambda = \frac{1}{2}$ in the identity (1.3.3), we get our parallelogram identity.*

Bessel Inequality(see e.g: [39])

Let $(X, \langle ., . \rangle)$ be an inner product space. Let $x \in X$ be arbitrary. If $\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence in X, then

$$\sum_{n=1}^{\infty} | \langle x, e_n \rangle |^2 \leq ||x||^2 \tag{1.3.4}$$

$\langle x, e_n \rangle$ is called the Fourier coefficient of x with respect to the orthonormal sequence $\{e_n\}_{n=1}^{\infty}$

1.4 Weak and Strong Convergence

1.4.1 Norm (strong) convergence

Let X be a normed linear space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Then $\{x_n\}_{n=1}^{\infty}$ is said to converge strongly (in norm) to a point $x \in X$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$, if given any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $n \geq \mathbb{N}_\epsilon$

$$||x_n - x|| < \epsilon \tag{1.4.1}$$

1.4.2 Weak convergence

Let X be a normed linear space over the scalar field $\mathbb{R}$, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Let $X^* = \{g : X \to \mathbb{R} | g \textit{ is linear and bounded}\}$, where X^* is called the dual of X. Then a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said to converge weakly to a point $x \in X$ and we write $x_n \rightharpoonup x$, if and only if $\forall\ f \in X^*$, we have $f(x_n) \to f(x)$.

Proposition 1.4.1 *Let H be a Hilbert space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in H. Then the sequence $\{x_n\}_{n=1}^{\infty}$ is said to converge weakly to x in H if and only if* $\forall$ z $\in$ H

$$\lim_{n\to\infty} \langle x_n, z\rangle = \langle x, z\rangle$$

Proof

($\Rightarrow$) Suppose $x_n \rightharpoonup x$, then $f(x_n) \to f(x)$ for all $f \in H^*$. Let $z \in H$ be arbitrary (but fixed). Then define $g : H \to \mathbb{R}$ by $g(x) = \langle x, z\rangle$.

Then g is linear since for all $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} g(\alpha x + \beta y) &= \langle \alpha x + \beta y, z\rangle \\ &= \langle \alpha x, z\rangle + \langle \beta y, z\rangle \\ &= \alpha \langle x, z\rangle + \beta \langle y, z\rangle \\ &= \alpha g(x) + \beta g(y). \end{aligned}$$

Also $\|g(x)\| = |\langle x, z\rangle| \leq \|x\|\|z\|$. This implies that g is bounded and $\|g\| \leq \|z\|$.
Thus $g \in H^*$. Since $g \in H^*$, and $x_n \rightharpoonup x$, then

$$g(x_n) \to g(x) \Rightarrow \langle x_n, z\rangle = \langle x, z\rangle .$$

($\Leftarrow$) Suppose $\langle x_n, z\rangle \to \langle x, z\rangle$, for all $z \in H$, then we prove that $x_n \rightharpoonup x$. Let $f \in H^*$ be arbitrary. Then by the Riez representation theorem, $\exists!\ z \in H$ such that $f(x) = \langle x, z\rangle$ for all $x \in H$.
Thus, $f(x_n) = \langle x_n, z\rangle$ and since $\langle x_n, z\rangle \rightharpoonup x$, we have that $f(x_n) \to f(x)$, which implies that $x_n \rightharpoonup x$.

1.4.3 Some Fundamental Results on Weak and Strong Convergence

Theorem 1.4.2 *Let $\{x_n\}_{n=1}^{\infty}$ be a weakly convergent sequence in a normed linear space X, then*

1. *The weak limit x of $\{x_n\}_{n=1}^{\infty}$ is unique*

2. *Every subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges weakly to x*

3. *The sequence $\{x_n\}_{n=1}^{\infty}$ is bounded*

Proof
(a) Suppose the sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to x and also to y, we show that $x = y$.
Now observe that weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x and to y implies that $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$ as well as $f(y)$ $\forall\ f \in X^*$ by definition. But $\{f(x_n)\}_{n=1}^{\infty}$ has unique limit

and hence

$$\begin{aligned}
f(x) &= f(y) \quad \forall\, f \in X^* \\
\Rightarrow f(x-y) &= 0 \qquad f \in X^* \\
\Rightarrow x-y &= 0, \qquad \text{(by a consequence of Hahn Banach Theorem)} \\
\Rightarrow x &= y
\end{aligned}$$

(b) Let $\{x_n\}_{n=1}^{\infty}$ converge weakly to x *in* X. We show that every subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges weakly to x. But $\{x_n\}_{n=1}^{\infty}$ converges weakly to x if and only if $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x) \in \mathbb{R}$ forall $f \in X^*$; if and only if every subsequence of $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x) \in \mathbb{R}\ \forall\, f \in X^*$; if and only if $\{f(x_{n_j})\}_{j=1}^{\infty}$ converges to $f(x)$ $\in \mathbb{R}\ \forall\, f \in X^*$; if and only if $\{x_{n_j}\}_{j=1}^{\infty}$ of converges to x in X.

(c) Since $\{x_n\}_{n=1}^{\infty}$ converges weakly to x, then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)\ \forall\, f \in X^*$, hence $\{f(x_n)\}_{n=1}^{\infty}$ is bounded $\forall\, f \in X^*$. Thus there exists $M_f > 0$ such that

$$|f(x_n)| \leq M_f \quad n \geq 1$$

Define $g_n : X^* \to \mathbb{R}$ by $g_n(f) = f(x_n)$

Then g_n is linear and bounded. Then by Uniformly boundedness theorem, we have that there exists $K > 0$ such that

$$||g_n|| \leq K \quad n \geq 1$$

but

$$\begin{aligned}
||g_n|| &= \sup\{|f(x_n)|, \quad f \in X^*, ||f|| = 1\} \\
&= ||x_n|| \leq K \qquad \text{[by a consequence of Hahn Banach Theorem]} \\
i.e., ||x_n|| &\leq K \quad \forall\, n \geq 1
\end{aligned}$$

Hence, the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. That is, a weakly convergent sequence is bounded.

Theorem 1.4.3 *Let X be a normed space over a scalar field $\mathbb{R}$. Then*

1. *Strong convergence of the sequence $\{x_n\}_{n=1}^{\infty}$ implies weak convergence of the sequence $\{x_n\}_{n=1}^{\infty}$*

2. *Weak convergence of the sequence $\{x_n\}_{n=1}^{\infty}$ does not in general imply strong convergence of the sequence $\{x_n\}_{n=1}^{\infty}$*

3. *In finite dimensional spaces, strong and weak convergence are the same*

Proof:

(1) Let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary sequence in X which converges strongly to $x \in X$, that is, $||x_n - x|| \to 0,$ as n $\to \infty$. We show that strong converges of $\{x_n\}_{n=1}^{\infty}$ to x implies weak convergence. Now, $\forall\ f \in X^*$, we have (using the linearity and boundedness of f)that

$$\begin{aligned} |f(x_n) - f(x)| &= |f(x_n - x)| \\ &\leq ||f||||x_n - x|| \\ &\to 0 \; as \; n \to \infty. \end{aligned}$$

This implies that $f(x_n) \to f(x)$ as $n \to \infty \quad \forall\ f \in X^*$.

Hence, $\{x_n\}_{n=1}^{\infty}$ converges weakly to x.

(2) Let $X = l_2(\mathbb{C}) = \{z = \{z_j\}_{j=1}^{\infty} : z_j \in C,\ j = 1, 2, 3, ..., \sum_{j=1}^{\infty} |z_j|^2 < \infty\}$. Define $\langle .,. \rangle : X \times X \to \mathbb{C} \quad \forall\ z, w \in X$ by

$$\langle z, w\rangle = \sum_{j=1}^{\infty} z_j \overline{w}_j.$$

Consider the sequence $\{x_n\}_{n=1}^{\infty} \subset l_2$ given by $x_n = (0, 0, 0, \cdots, 1, 0, 0, 0, \cdots) = e_n$ where 1 is in the nth position. Then the sequence $\{x_n\}_{n=1}^{\infty}$ is an orthonormal sequence in l_2. Let $f \in X^*$ be arbitrary. Then since X is a Hilbert space and f is bounded linear functional on X, we have that $\forall\ x \in X$ there exists unique $z \in X$ such that,

i $f(x) = \langle x, z\rangle$

ii $\|f\| = \|z\|$

Hence, $f(x_n) = f(e_n) = \langle e_n, z\rangle,\;\; n \geq 1$

by Bessel Inequality, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\,\langle e_n, z\rangle\,|^2 &\leq \|z\|^2 \\
&\Rightarrow \sum_{n=1}^{\infty} |\,\langle e_n, z\rangle\,|^2 \text{ converges and } \lim_{\mathrm{n}\to\infty} |\,\langle \mathrm{e_n}, \mathrm{z}\rangle\,|^2 = 0 \\
&\Rightarrow \lim_{n\to\infty} |\,\langle e_n, z\rangle\,| = 0 \\
&\Rightarrow \lim_{n\to\infty} f(x_n) = 0 = f(0) \quad \forall\, f \in X^* \\
&\Rightarrow x_n \to 0 \;\text{ as n} \to \infty
\end{aligned}$$

However, the sequence $\{x_n\}_{n=1}^{\infty}$ is not Cauchy, and hence not strongly convergent since for $m \neq n$, we have $||x_n - x_m|| = \sqrt{2}$ which does not go to zero as $n, m \to \infty$

(3) Let X be finite dimensional say of dimension m. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X which converges weakly to $x \in X$. Since X is m dimensional, let $b = (v_1, v_2, v_3, ..., v_m)$ be any given basis for X. Then

$$\begin{aligned}
x &= \sum_{i=1}^{m} \alpha_i v_i \qquad \alpha_i \;\text{ scalars} \\
x_n &= \sum_{i=1}^{m} \alpha_i^n v_i \qquad \alpha_i^n \;\text{ scalars}
\end{aligned}$$

$\{\mathrm{x_n}\}_{\mathrm{n}=1}^{\infty}$ converges weakly to x implies that $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$ $\forall\,\mathrm{f} \in \mathrm{X}^*$, and so

$$\begin{aligned}
f(x) &= \sum_{i=1}^{m} \alpha_i f(v_i) \qquad \alpha_i \;\text{ scalars} \\
f(x_n) &= \sum_{i=1}^{m} \alpha_i^n f(v_i) \qquad \alpha_i^n \;\text{ scalars}
\end{aligned}$$

In particular, we may consider $f_1, f_2, f_3, \ldots, f_n$ defined by $f_j(v_j) = 1,\; f_j(v_i) = 0 \;\, i \neq j$. Since $f_j(x_n) \to f_j(x)$ we must have $\alpha_j^n \to \alpha_j \; as \; n \to \infty$, that is $|\alpha_j^n - \alpha_j| \to 0 \; as \; n \to \infty$

$\forall \quad j = 1, 2, 3, \ldots, m$. Observe

$$\begin{aligned}
\|x_n - x\| &= \|\sum_{j=1}^{m} \alpha_j^n v_j - \sum_{j=1}^{m} \alpha_j v_j\| \\
&= \|\sum_{j=1}^{m} (\alpha_j^n - \alpha_j) v_j\| \\
&\leq \sum_{j=1}^{m} |\alpha_j^n - \alpha_j| \|v_j\| \\
&\to 0 \quad as\ n \to \infty, \quad \text{that is} \\
\|x_n - x\| &\to 0 \quad \text{as n} \to \infty, \\
&\Rightarrow x_n \to x \quad \text{as n} \to \infty
\end{aligned}$$

1.5 Nearest Point Theorem

Theorem 1.5.1 *Let $(X, \langle ., . \rangle)$ be an inner product space and C a nonempty convex and complete subset of X. Then for all $x \in X$, there exist a unique $y^* \in C$ such that $\delta = \|x - y^*\| = \inf_{y \in C} \|x - y\|$. If C is a nonempty complete subspace, then $(x - y^*) \perp C$.*

Proof of Theorem 1.5.1

Since $\delta = \inf_{y \in C} \|x - y\|$, then there exists a sequence $\{c_n\}_{n=1}^{\infty} \subseteq C$ such that $\lim_{n \to \infty} ||x - c_n|| = \delta$. We proceed to show that $\{c_n\}_{n=1}^{\infty}$ is Cauchy. That is, $||c_n - c_m|| \to 0$ as n, m $\to \infty$. Now, for each $x \in X$,

$$\begin{aligned}
||c_n - c_m||^2 &= ||c_n - x + x - c_m||^2 \\
&= ||x - c_m - (x - c_n)||^2 \\
&= 2||x - c_m||^2 + 2||x - c_n||^2 - ||2x - (c_n + c_m)||^2 \qquad (1)
\end{aligned}$$

by parallelogram's identity. Also

$$\begin{aligned}
||2x - (c_n + c_m)||^2 &= 4||x - \frac{1}{2}(c_n + c_m)||^2 \\
&\geq 4\delta^2 \qquad (2)
\end{aligned}$$

using (2) in (1), we have

$$\begin{aligned} ||c_n - c_m||^2 &= 2||x - c_m||^2 + 2||x - c_n||^2 - ||2x - (c_n + c_m)||^2 \\ &= 2||x - c_m||^2 + 2||x - c_n||^2 - 4\delta^2 \end{aligned}$$

Therefore, $\lim\limits_{n,m\to\infty} \|c_n - c_m\|^2 = 0$. Hence, $\{c_n\}_{n=1}^{\infty}$ is Cauchy. Again since $\{c_n\}_{n=1}^{\infty} \subseteq C$, and C is complete, then $c_n \to y^* \in C$. Also since $y^* \in C$, we must have that $\|x - y^*\| \geq \delta$ Observe that

$$\delta \leq \|x - y^*\| \leq \|x - c_n\| + \|c_n - y^*\|.$$

Letting $n \to \infty$, we obtain

$$\delta \leq \|x - y^*\| \leq \delta.$$

Thus, $\|x - y^*\| = \delta = \inf\limits_{y\in C} \|x - y\|$.

We proceed to show that y^* is unique. To see this, suppose there exists another $y_0 \in C$ such that $\|x - y_0\| = \delta$; we show that $y^* = y_0$. Thus

$$\begin{aligned} 0 &\leq \|y^* - y_0\|^2 = \|y^* - x + x - y_0\|^2 \\ &= \|y^* - x - (y_0 - x)\|^2 \\ &= 2(\|x - y^*\|^2 + \|x - y_0\|^2) - \|2x - (y^* + y_0)\|^2 \quad \text{(using parallelogram identity)} \\ &= 2(\|x - y^*\|^2 + \|x - y_0\|^2) - 4\|x - \frac{1}{2}(y^* + y_0)\|^2 \end{aligned}$$

Thus,

$$\|y^* - y_0\|^2 \leq 2(\delta^2 + \delta^2) - 4\delta^2 = 0.$$

This implies that

$0 \leq \|y^* - y_0\|^2 \leq 0$ and so $y^* = y_0$.

Corollary 1.5.2 *Let H be a Hilbert space, and let C a nonempty convex closed subset of H. Then for all $x \in H$, there exists a unique $y^* \in C$ such that $\delta = \|x - y^*\| = \inf\limits_{y\in C} \|x - y\|$. If C is a closed subspace, then $(x - y^*) \perp C$.*

Definition 1.5.3 *Let H be a Hilbert space and C a nonempty closed convex subset of H. A mapping $P_C : H \to C$ is called projection of H unto C if $\forall\, x \in H, P_C(x) \in C$ and*

$$||x - P_C(x)|| = \inf\{||x - y||, \quad y \in C\} \tag{1.5.1}$$

In other words, this map assigns every point in H to its unique nearest point in C.

Lemma 1.5.4 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $P_C : H \to C$ be the projection of H onto C. Then, P_C is characterized by the following relations;*

(i) $for\ x \in H$, $x^* \in C$ satisfies $x^* = P_C(x) \Leftrightarrow \langle x - x^*, y - x^*\rangle \leq 0$ for all $y \in C$

(ii) P_C is firmly nonexpansive, that is, $||P_C(x) - P_C(y)||^2 \leq \langle x - y, P_C(x) - P_C(y)\rangle$, $\forall$ x, y $\in$ H

(iii) P_C is nonexpansive, that is, $||P_C(x) - P_C(y)|| \leq ||x - y||$, $\forall$ x, y $\in$ H

(iv) Restriction of P_C to C is identity maping of C onto C (i.e., $P_C|_C = I$)

(v) P_C is idempotent on H.

Proof of Lemma 1.2

Proof of (i): Observe that $\|x - P_C(x)\| \leq \|x - y\|$, for all $y \in C$

Let $y = (1 - \lambda)P_C(x) + \lambda u, \lambda \in (0, 1), u \in C$. Then, $(1 - \lambda)P_C(x) + \lambda u \in C$, for all $u \in C$. Furthermore,

$$\begin{aligned}
\|x - P_C(x)\|^2 &\leq \|x - [(1 - \lambda)P_C(x) + \lambda u]\|^2 \\
&= \|x - P_C(x) + \lambda(P_C(x) - u)\|^2 \\
&= \|x - P_C(x)\|^2 + 2\lambda\langle x - P_C(x), P_C(x) - u\rangle + \lambda^2\|P_C(x) - u\|^2.
\end{aligned}$$

Thus, $\langle x - P_C(x), u - P_C(x)\rangle \leq \lambda\|P_C(x) - u\|^2$.

Letting $\lambda \to 0$, we obtain

$$\langle x - P_C(x), u - P_C(x)\rangle \leq 0, \quad \text{for all u} \in \text{ C}.$$

conversely, if for some $z \in C$, we have the property $\langle x-P_C(x), z-P_C(x)\rangle \leq 0$, for all y $\in$ C, then $z = P_C(x)$.

To see this, observe that

$$\begin{aligned}
\|x-y\|^2 &= \|x-z+z-y\|^2 \\
&= \|x-z\|^2 - 2\lambda\langle x-z, y-z\rangle + \lambda^2\|z-y\|^2 \\
\|x-y\|^2 &\geq \|x-z\|^2 \;\; y \in C
\end{aligned}$$

Thus by corollary 1.5.2, $z = P_C(x)$.

Proof of (ii): Let $x \in H$, then we obtain from (i) that $z = P_C(x)$ if and only if $\langle x - z, z-y\rangle \geq 0$, for all $y \in C$. In particular,

$$\langle x - P_C x, P_C(x) - P_C(y)\rangle \geq 0 \tag{1.5.2}$$

In a similar argument, $z = P_C(y)$ if and only if $\langle y - P_C(y), P_C(y) - P_C(x)\rangle \geq 0$, for all $x \in C$. That is,

$$\langle P_C(y) - y, P_C(x) - P_C(y)\rangle \geq 0 \tag{1.5.3}$$

Adding equation (1.5.3) to equation (1.5.2) yields

$$\langle x - P_C(x) + P_C(y) - y, P_C(x) - P_C(y)\rangle \geq 0.$$

This implies that

$$\langle x - y, P_C(x) - P_C(y)\rangle - \langle P_C(x) - P_C(y), P_C(x) - P_C(y)\rangle \geq 0$$

and so

$$\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2. \tag{1.5.4}$$

This shows that P_C is firmly nonexpansive.

Proof of (iii): by Cauchy-Schwartz inequality, we obtain from (1.5.4) that

$$\begin{aligned}
\|P_C(x) - P_C(y)\|^2 &\leq \langle x - y, P_C(x) - P_C(y)\rangle \\
&\leq |\,\langle x - y, P_C(x) - P_C(y)\rangle\,| \\
&\leq \|x - y\|\|P_C(x) - P_C(y)\|.
\end{aligned}$$

dividing through by $\|P_C(x) - P_C(y)\| \neq 0$, It follows that,

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|, \quad \forall\ x, y \in H$$

Proof of (iv): Let $x \in H$, we have that

$$\|x - P_C(x)\| = \inf_{y \in C} \|x - y\| \leq \|x - y\|, \quad \forall\ y \in C. \tag{1.5.5}$$

If $x \in C$, and in particular if $y = x$, then equation (1.5.5) becomes

$$\|x - P_C x\| = 0.$$

Thus, $P_C(x) = x$ and this implies that P_C is an identity on C.

(v) Let $x \in H$, and let $P_C(x) = z$. Then,

$$P_C^2(x) = P_C(P_C(x)) = P_C(z) = z,$$

since $z \in C$. Therefore $P_C^2(x) = P_C(x)$ and so $P_C^2 = P_C$.

1.6 Reflexive Spaces

1.6.1 Canonical Mapping

Definition 1.6.1 *Let X be a real normed linear space over the scalar field $\mathbb{R}$. Let X^* be the dual of X and $(X^*)^*$ the double dual. The map $U : X \to (X^*)^*$ defined by*

$$U(x) = \phi_x \quad \forall\ x \in X$$

(where $\phi_x : X^* \to \mathbb{R}$ is a bounded linear mappin with domain X^* and range $\mathbb{R}$ (scalar field) defined by

$$\phi_x(f) = f(x) \quad \forall\ f \in X^*)$$

is called the canonical map.

Properties of U:

U has the following important and useful properties

1. U is linear

   To see this, take arbitrary $x, y \in X$ and scalars α, β then we have that

$$U(\alpha x + \beta y) = \phi_{\alpha x + \beta y} \qquad (i)$$

   But

$$\begin{aligned}
\phi_{\alpha x+\beta y}(f) &= f(\alpha x + \beta y) \quad \forall\, f \in X^* \\
&= \alpha f(x) + \beta f(y) \quad \forall\, f \in X^* \\
&= \alpha \phi_x(f) + \beta \phi_y(f) \quad \forall\, f \in X^* \\
&= (\alpha \phi_x + \beta \phi_y)(f) \quad \forall\, f \in X^*. \\
So,\ \phi_{\alpha x+\beta y}(f) &= \alpha \phi_x + \beta \phi_y \\
&\iff U(\alpha x + \beta y) = \alpha U(x) + \beta U(y)
\end{aligned}$$

   Thus, $U(\alpha x + \beta y) = \alpha U(x) + \beta U(y)\ \ x,\ y \in X$

2. U is one to one since if

$$\begin{aligned}
U(x) &= U(y) \quad then \\
\phi_x(f) &= \phi_y(f) \quad \forall\, f \in X^* \\
\Rightarrow f(x) &= f(y) \quad \forall\, f \in X^* \\
\Rightarrow f(x) - f(y) &= 0 \quad \forall\, f \in X^* \\
\Rightarrow f(x - y) &= 0 \quad \forall\, f \in X^* \\
\Rightarrow x - y &= 0 \quad \text{(by a consequence of Hahn Banach Theorem)} \\
\Rightarrow x &= y
\end{aligned}$$

3. U is bounded

$$\begin{aligned}
||U|| &= \sup\{|\phi_x(f)| : \|f\| = 1, f \in X^*\} \\
&= \sup\{|f(x)|, ||f|| = 1, f \in X^*\} \\
&= ||x|| \text{ (by a consequence of Hahn Banach Theorem)} \\
\text{Thus,}\quad ||U|| &= ||x||
\end{aligned}$$

This implies that U is bounded.

4. U is isometry. This is true since

$$\begin{aligned} ||U(x)-U(y)|| &= ||U(x-y)|| \qquad \text{(U is linear)} \\ &= ||x-y|| \qquad \text{(by 3 above)} \end{aligned}$$

Remark 1.6.2 *We remark that in general U is not onto.*

Definition 1.6.3 *Let X be a normed linear space and let U be the canonical map of X into $(X^*)^*$. If U is onto, then X is called reflexive. In other words, a reflexive space is one in which the canonical map is onto. In this case, $U(X) = (X^*)^*$*

Definition 1.6.4 *A normed linear space X is uniformly convex if $\forall\ \epsilon > 0,\ \exists\ \delta = \delta(\epsilon) > 0$ such that forall $x, y \in X$*

$$\|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \epsilon \implies \|\frac{x+y}{2}\| \leq 1-\delta.$$

Proposition 1.6.5 *Every Hilbert space is Uniformly Convex.*

Proof: It suffices to show that every Hilbert space H is uniformly convex. And by Milman-Pettis Theorem we conclude that every Hilbert space H is reflexive. Now, for each $x,\ y \in H$

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2) \qquad \text{parallelogram identity} \qquad (1)$$

Let $\epsilon \in (0, 2]$ be given and let $x,\ y \in H$ be such that $||x|| \leq 1,\ ||y|| \leq 1$ and $||x-y|| \geq \epsilon$. Then (1) yields,

$$\begin{aligned} \|x+y\|^2 &= 2(\|x\|^2 + \|y\|^2) - \|x-y\|^2 \\ \frac{1}{4}\|x+y\|^2 &= \frac{1}{2}(\|x\|^2 + \|y\|^2) - \frac{1}{4}\|x-y\|^2 \end{aligned}$$

$$
\begin{aligned}
\|\frac{1}{2}(x+y)\|^2 &= \frac{1}{2}(\|x\|^2+\|y\|^2) - \frac{1}{4}\|x-y\|^2 \\
&\leq 1 - \frac{1}{4}||x-y||^2 \\
\text{Thus, } \|\frac{1}{2}(x+y)\|^2 &\leq 1 - \frac{1}{4}\epsilon^2 \\
\text{so that, } \|\frac{1}{2}(x+y)\| &\leq \sqrt{1-\frac{1}{4}\epsilon^2}
\end{aligned}
$$

Choose $\delta = \sqrt{1-\frac{1}{4}\epsilon^2}$. Thus, every Hilbert space, H is uniformly convex.

Example 1.6.6 *Every finite-dimensional normed space is reflexive.*

Example 1.6.7 l_1 *and* l_∞ *with the 'sup norm', respectively, are not reflexive.*

Example 1.6.8 *The Banach space* $C[a,b]$ *of continuous functions on* $[a,b]$ *with the 'sup norm' is not reflexive.*

Theorem 1.6.9 (Milman-Pettis) *Every uniformly convex Banach space is reflexive.*

Particularly, every Hilbert space is reflexive.

Chapter 2

SOME IMPORTANT CLASSES OF OPERATORS AND ITERATIVE PROCESSES

2.1 Definitions

Definition 2.1.1 *A function $\Phi : [0, \infty) \to [0, \infty)$ is said to be a guage function if Φ is continuous and strictly increasing with $\Phi(0) = 0$ and $\lim\limits_{t\to\infty} \Phi(t) = \infty$.*

Definition 2.1.2 *Let E be a real Banach space. Let E^* denote the dual of E and 2^{E^*} be the collection of all subsets of E^*. Let $\Phi : [0, \infty) \to [0, \infty)$ be a guage function. The mapping J_Φ:$E \to 2^{E^*}$ defined by*

$$J_\Phi \mathrm{X} = \{f \in E^* : \langle \mathrm{X}, f\rangle = \|x\|\|f\|, \|f\| = \Phi(\|x\|)\}$$

is called duality map with gauge function Φ, where $\langle ., .\rangle$ denotes the duality pairing between E and E^*.

We note that if $1 < q < \infty$, then $\Phi(t) = t^{q-1}$ is a gauge function. The duality mapping $J_q : E \to 2^{E^*}$ with gauge $\Phi(t) = t^{q-1}$ defined for each $x \in E$ by

$$J_\Phi(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|^{q-1}\}$$

is called the *generalised duality mapping.* If $q = 2$, we obtain

$$J_2 := J : E \to 2^{E^*}$$

defined for all $x \in E$ by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|\}.$$

J is known as the normalised duality map.

Lemma 2.1(Kato,[17]): Let E be an arbitrary Banach space over the scalar field $\mathbb{R}$. Then the following are equivalent:

1. if $x, y \in E$ are such that

$$\|x\| \leq \|x + \lambda y\|, \ \lambda > 0,$$

   then there exists $j(x) \in J(x)$ such that $Re\langle y, j(x)\rangle \geq 0$.

2. if for $x \in E$ there exists $j(x) \in J(x)$ such that $Re\langle y, j(x)\rangle \geq 0, \ y \in E,$ then

$$\|x\| \leq \|x + \lambda y\|, \ \forall \ \lambda > 0,$$

2.2 OPERATORS

2.2.1 Lipschitz operators:

Let E be an arbitrary normed Linear space and let C be a nonempty subset of E. A mapping $T : C \to C$ is said to be *Lipschitz continuous* if $\exists$ a constant $L \geq 0$ such that for all $x, \ y \in C$,

$$\|Tx - Ty\| \leq L\|x - y\| \tag{2.2.1}$$

If $L \in (0, 1)$, then T is called a *contraction.* The mapping T is said to be *nonexpansive* if $L = 1$. Nonepansive mappings are linked intimately with several other nonlinear mappings that are of interest in ordinary and partial differential equations (for example $[2, 3, 4, 7, 11]$). Bruck [6] remarked that the intimate connection between nonexpansive operators and *accretive* operators (which we define next in the sequel) accounts partly for the importance of nonexpansive mappings. The class of nonexpansive mapping is one of the initial classes of operators for which fixed point results were obtained.

2.2.2 Accretive Operators

Let E be a Banach space. An operator $T : D(T) \subseteq E \to E$ is said to be *accretive* if $\forall\, x,\ y \in D(T)$ and $\lambda > 0$,

$$\|x - y\| \leq \|x - y + \lambda(Tx - Ty)\|. \tag{2.2.2}$$

By Lemma (2.1) we obtain that (2.2.2) is equivalent to the statement: there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq 0. \tag{2.2.3}$$

In Hilbert spaces, (2.2.3) reduces to

$$\langle Tx - Ty, (x - y)\rangle \geq 0 \tag{2.2.4}$$

for all $x, y \in D(T)$, and T is called a *monotone* operator in this case. Let C be a nonempty subset of a real Hilbert space. A mapping $T : C \to C$ is called *η-strongly monotone* if and only if there exists $\eta > 0$ such that for all $x, y \in C$,

$$\langle Tx - Ty, (x - y)\rangle \geq \eta\|x - y\|^2. \tag{2.2.5}$$

2.2.3 Strictly Pseudocontractive Mappings

Let E be a real Banach space and C a nonempty closed convex subset of E . A mapping T with domain $D(T)$ and range $R(T)$, both subsets of C, is called a *strictly pseudocontractive* mapping if and only if there exist $\lambda > 0$ such that for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \lambda\|x - y - (Tx - Ty)\|^2 \tag{2.2.6}$$

Without loss of generality we may assume $\lambda \in (0, 1)$. If I denotes the identity mapping , then (2.2.6) can be written as

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \lambda\|(I - T)x - (I - T)y\|^2 \tag{2.2.7}$$

In Hilbert spaces (2.2.6) and (2.2.7) are equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \tag{2.2.8}$$

with $k = (1 - 2\lambda) < 1$. We can assume also $k \geq 0$, so that $k \in [0, 1)$. It is shown in [32] that a strictly pseudocontractive mapping is $L-$ Lipschitz. In Hilbert spaces

$$\|Tx - Ty\| \leq \frac{1 + \sqrt{k}}{1 - \sqrt{k}}\|x - y\| = L\|x - y\| \tag{2.2.9}$$

for all $x, y \in D(T),\ L > 0$, while in arbitrary Banach spaces we have

$$\|Tx - Ty\| \leq \frac{1 + \lambda}{\lambda}\|x - y\| = L\|x - y\| \tag{2.2.10}$$

for all $x, y \in D(T)$.

2.2.4 Pseudocontractive Mappings

Let E be a real Banach space and C a nonempty subset of E. A mapping $T : C \to C$ is said to be *pseudocontractive* if for all $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 \tag{2.2.11}$$

If I denotes the identity operator from C to C, then (2.2.11) is equivalent to

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq 0 \tag{2.2.12}$$

Thus T is pseudocontractive if and only if $(I - T)$ is accretive.

It was shown by Browder and Petryshyn [5] that in Hilbert spaces, (2.2.11) and (2.2.12) reduce to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \tag{2.2.13}$$

Remark 2.2.1 *The following example shows that the class of pseudocontractive mappings properly contains the class of nonexpansive mappings. It also shows that the class of pseudocontractive mappings is more general than the class of strictly pseudocontractive mappings.*

Example 2.2.2 *Let $\mathbb{R}$ be the reals with the usual norm and $C = [0,1]$. Let*

$$T : [0,1] \to [0,1]$$

be defined by $Tx = 1 - x^{\frac{2}{3}}$.

claim: *T is not Lipschitz.*

proof of claim:

To see this, let $L > 0$ be arbitrary, $r = \min\{1, \frac{1}{L^3}\}$, and consider $x \in (0, r]$,

$y = 0$. Then $|x - y| = |x| = x$ *and*

$$|Tx - Ty| \;\;=\;\; |1 - x^{\frac{2}{3}} - 1| = x^{\frac{2}{3}} = x^{\frac{-1}{3}}(x)$$

Since $x < \frac{1}{L^3}$, then $x^{\frac{1}{3}} < \frac{1}{L}$. Thus $\frac{1}{x^{\frac{1}{3}}} > L$ (i.e. $x^{\frac{-1}{3}} > L$). Thus

$$|Tx - Ty| = x^{\frac{-1}{3}}(x) > Lx = L|x| = L|x - y|.$$

establishing our claim. Since T is not Lipschitz, it is not nonexpansive. Let $x, y, \in [0,1]$, and let $x \leq y$. Then

$x^{\frac{2}{3}} \leq y^{\frac{2}{3}}$ and $-x^{\frac{2}{3}} \geq -y^{\frac{2}{3}}$. Thus $1 - x^{\frac{2}{3}} \geq 1 - y^{\frac{2}{3}}$, and this yields $Tx \geq Ty$.

It follows that if

$$x - y \leq 0, \text{ then } Tx - Ty \geq 0.$$

Hence

$$\langle Tx - Ty, x - y \rangle \leq 0 \leq t|x - y|^2 \;\; \forall\, t > 0$$

Thus T is pseudocontractive. Observe that the mapping T is not strictly pseudocontractive because it is not Lipschitz.

2.2.5 Demicontractive Mappings

Let E be a real Banach space and C a nonempty subset of E. A mapping $T : C \to$C is said to be *demicontractive* if $F(T) = \{x \in C : Tx = x\} \neq \emptyset$ and for all $x \in C,\ p \in F(T)$, there exist $j(x - y) \in J(x - y)$ and a constant $\lambda > 0$ such that

$$\langle x - Tx, j(x - p) \rangle \geq \lambda \|x - Tx\|^2. \tag{2.2.14}$$

If we set $\lambda = \frac{1}{2}(1-k)$, $k \in [0,1)$. Then (2.2.14) becomes

$$\langle x - Tx, j(x-p)\rangle \geq \frac{(1-k)}{2}\|x - Tx\|^2 \tag{2.2.15}$$

In Hilbert spaces, (2.2.14) and (2.2.15) are equivalent to

$$||Tx - p||^2 \leq ||x-p||^2 + k||x - Tx||^2. \tag{2.2.16}$$

Hicks and Kubicek [13] in 1977 and Maruster [22] in 1977 introduced this class of mappings independently. In real Hilbert spaces H, Hicks and Kubicek called the mapping $T : D(T) \subseteq H \to C$ demicontractive if (2.2.16) is satified. Maruster called $T : D(T) \subseteq H \to C$ an operator satisfying condition (A) if (2.2.14) is satified.

Remark 2.2.3 *If a mapping T is strictly pseudocontractive with $F(T) \neq \emptyset$, then T is demicontractive mapping. Below is an example of a demicontractive mapping which is not pseudocontractive and hence not strictly pseudocontractive.*

Example 2.2 (Hicks and Cubicek [13]

Let $\mathbb{R}$ be the real line with the usual norm. Let $T : [-1,1] \to \mathbb{R}$ be definded by

$$Tx = \begin{cases} \frac{2}{3}x\sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Clearly, 0 is the only fixed point of T. Also for $x \in [0,1]$, $|Tx - 0|^2 = |Tx|^2 = |\frac{2}{3}x\sin(\frac{1}{x})|^2 \leq |\frac{2}{3}x|^2 \leq |x|^2 \leq |x-0|^2 + \lambda|Tx - x|^2$ for any $\lambda < 1$. Thus, T is demicontractive.

T is not pseudocontractive. For if $x = \frac{2}{\pi}$, $y = \frac{2}{3\pi}$, then $|Tx - Ty|^2 = \frac{256}{81\pi^2}$.

However,

$$|x-y|^2 + |(I-T)x - (I-T)y|^2 = \frac{160}{81\pi^2}$$

Thus,

$$\begin{aligned} |Tx - Ty|^2 &> |x \ - y|^2 + |(I-T)x - (I-T)y|^2 \\ &> |x-y|^2 + k|(I-T)x - (I-T)y|^2 \end{aligned}$$

for all $\lambda \in [0,1)$, so that T is not pseudocontractive and therefore not strictly pseudocontrative.

2.2.6 Asymptotically Nonexpansive operators:

Let E be a Normed space and let C be a nonempty subset of E. Let C be a nonempty subset of a normed space E. A mapping $T : C \to C$ is called *asymptotically nonexpansive* if and only if $\forall\, x, y \in C, \exists$ a sequence $\{k_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq\; k_n\|x - y\| \tag{2.2.17}$$

2.2.7 Uniformly L-Lipschitzian Mappings:

Let E be a Normed space and let C be a nonempty subset of E. Let $T : C \to C$ be a mapping (possibly nonlinear) from C *into* C. The mapping T is called Uniformly L-Lipschitzian if and only if there is a constant $L > 0$ such that $\forall\, x, y \in C$ and $n \geq 1$

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{2.2.18}$$

2.2.8 k-strictly Asymptotically Pseudocontractive operators:

Let E be a Normed linear space. Let C be a nonempty subset of E. A mapping $T : C \to C$ is called *k-strictly asymptotically pseudocontractive* if and only if there exists a constant $k \in [0, 1)$ and a sequence $\{k_n\}_{n=1}^{\infty}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq \frac{1}{2}(1-k)\|(I-T^n)x-(I-T^n)y\|^2-\frac{1}{2}(k_n^2-1)\|x-y\|^2 \tag{2.2.19}$$

for all $n \in N$.

2.2.9 Asymptotically demicontractive operators:

Let E be a normed space and let C be a nonempty subset of E. A mapping $T : C \to C$ is called *asymptotically demicontractive* if and only if there exists a constant $k \in [0, 1)$ and a sequence $\{k_n\}_{n=1}^{\infty}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x \in C,\; p \in F(T)$, there exists $j(x-p) \in J(x-p)$ such that

$$\langle x - T^n x, j(x-p)\rangle \geq \frac{1}{2}(1-k)\|x - T^n x\|^2 - \frac{1}{2}(k_n^2 - 1)\|x-p\|^2 \tag{2.2.20}$$

In 1996, Osilike [27] proved that in Hilbert spaces, (2.2.17) and (2.2.18) reduces to

$$\|T^n x - T^n y\|^2 \leq k_n^2\|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2 \tag{2.2.21}$$

and

$$\|T^n x - p\|^2 \leq k_n^2\|x-p\|^2 + k\|x - T^n x\|^2 \tag{2.2.22}$$

Remark 2.2.4 *Let H be a real Hilbert space. Let C be a nonempty subset of H. The mapping $T : C \to C$ is asymptotically demicontractive, if and only if for all $x \in C$, $p \in F(T)$,*

$$2\langle x - T^n x, x - p\rangle \geq (1-k)||x - T^n x||^2 - (k_n^2 - 1)||x-p||^2 \tag{2.2.23}$$

Verification of (2.2.22):

Let T be asymptotically demicontractive, then for all $\forall\ x, y \in C$, $p \in F(T)$ we observe that

$$\begin{aligned} ||T^n x - p||^2 &= ||x - p - (x - T^n x)||^2 \\ &= ||x-p||^2 + ||x - T^n x||^2 - 2\langle x - T^n x, x - p\rangle \end{aligned} \tag{2.2.24}$$

Since T is asymptotically demicontractive,

$$\|x-p\|^2 + \|x - T^n x\|^2 - 2\langle x - T^n x, x-p\rangle \leq k_n^2\|x-p\|^2 + k\|x - T^n x\|^2,$$

so that

$$-2\langle x - T^n x, x-p\rangle \leq k_n^2\|x-p\|^2 + k\|x - T^n x\|^2 - \|x-p\|^2 - \|x - T^n x\|^2$$

diving through by (-1)

$$2\langle x - T^n x, x-p\rangle \geq -k_n^2\|x-p\|^2 - k\|x - T^n x\|^2 + \|x-p\|^2 + \|x - T^n x\|^2$$

rearranging yields:

$$2\langle x - T^n x, x - p\rangle \geq (1-k)||x - T^n x||^2 - (k_n^2 - 1)||x - p||^2 \tag{2.2.25}$$

On the other hand, using inequality (2.2.24) in (2.2.23), we obtain:

$$\begin{aligned} ||T^n x - p||^2 &\leq ||x-p||^2 - (1-k)||x - T^n x||^2 + ||x - T^n x||^2 + (k_n^2 - 1)||x-p||^2 \\ &= ||x-p||^2 - ||x - T^n x||^2 + k||x - T^n x||^2 + ||x - T^n x||^2 \\ &= +k_n^2||x-p||^2 - ||x-p||^2 \\ &= k_n^2||x-p||^2 + k||x - T^n x||^2 \end{aligned}$$

that is,

$$||T^n x - p||^2 \leq k_n^2||x-p||^2 + k||x - T^n x||^2 \tag{2.2.26}$$

Combining (2.2.24) and (2.2.25) establishes (2.2.22), that is,

$$2\langle x - T^n x, x - p\rangle \geq (1-k)||x - T^n x||^2 - (k_n^2 - 1)||x - p||^2$$

2.3 ITERATIVE PROCESSES

In this section we review some of the iterative processes.

2.3.1 The Picard Iteration Process

Let E be a linear space. Let $T : K \to K$ be a mapping, where K is a nonempty convex subset of E. The Picard iteration process [8] is given by

$$x_1 \in K, \quad x_{n+1} = Tx_n, \quad n \geq 1. \tag{2.3.1}$$

2.3.2 The Krasnoselskii Iteration Process

Let E be a linear space. Let $T : K \to K$ be a mapping, where K is a nonempty convex subset of E. The Krasnoselskii iteration process [8] is given by

$$x_0 \in K, \quad x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, \quad n \geq 0,\ \lambda \in (0,1) \tag{2.3.2}$$

2.3.3 The Mann Iteration Process

Let E be a linear space. Let $T : K \to K$ be a mapping, where K is a nonempty convex subset of E. The Mann iteration process [20] is given by

$$x_0 \in K, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \tag{2.3.3}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying the following conditions:

i $\lim\limits_{n\to\infty} \alpha_n = 0$

ii $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$

2.3.4 The Ishikawa Iteration Process

Let E be a linear space and let $T : K \to K$ be a mapping, where K is a nonempty convex subset of E. The Ishikawa iteration process [16] which is defined for an arbitrary $x_0 \in E$ is given by

$$\begin{aligned} x_0 \in K, \quad y_n &= (1-\beta_n)x_n + \beta_n T x_n \quad n \geq 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \quad n \geq 0, \end{aligned} \tag{2.3.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are suitable sequences in $[0, 1]$ satisfying the following conditions:

i $0 \leq \alpha_n \leq \beta_n \leq 1$

i $\lim\limits_{n\to\infty} \beta_n = 0$

ii $\sum\limits_{n=1}^{\infty} \alpha_n \beta_n = \infty$

In [35], Rhoades compared the performance of these two iteration processes and showed that despite the fact that they are similar, they may exhibit different behaviours for different classes of nonlinear mappings. In 1995, Liu [18] introduced the Ishikawa and Mann iteration processes with "errors" as follows:

2.3.5 Mann-type Iteration Process with Errors in the sense of Liu

Let E be a linear space and let $T : E \to E$ be a mapping. Then the Mann-type iteration process with errors in the sense of Liu is given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n + U_n, \quad n \geq 1 \tag{2.3.5}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a suitable real sequence in $[0, 1]$ and $\sum_{n=1}^{\infty} ||U_n|| < \infty$).

2.3.6 Ishikawa-type Iteration Process with Errors in the sense of Liu

Let E be a linear space and let $T : E \to E$ be a mapping. The Ishikawa-type iteration process with errors is given by

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n T x_n + U_n, \quad n \geq 0 \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n + V_n, \quad n \geq 0 \end{aligned} \tag{2.3.6}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are suitable sequences in $[0, 1]$ and $\{U_n\}_{n=0}^{\infty}$, $\{V_n\}_{n=0}^{\infty}$ are summable sequences in E.

Xu [38], introduced a more satisfactory definition of Ishikawa type and Mann type iteration processes with errors as follows.

2.3.7 Mann-type and Ishikawa-type Iteration Process with Errors in the sense of Xu

Let C be a nonempty convex subset of a Banach space, E, and $T : C \to C$ a mapping. For any given $x_0 \in C$,the sequence $\{x_n\}_{n=0}^{\infty}$ defined iteratively by

$$\begin{aligned} y_n &= a_n x_n + b_n T x_n + c_n U_n, \quad n \geq 0 \\ x_{n+1} &= a'_n x_n + b'_n T y_n + c'_n V_n \quad n \geq 0 \end{aligned} \tag{2.3.7}$$

where $\{U_n\}_{n=0}^{\infty}, \{V_n\}_{n=0}^{\infty}$ are bounded sequences in C and $\{a_n\}$,$\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are real sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ for all integers $n \geq 0$ is called the Mann-type and Ishikawa-type iteration process with errors in the sense of Xu. If in (2.3.5), $a_n = 1, b_n, c_n = 0$ for all $n \geq 0$, then $y_n = x_n$ and the sequence $\{x_n\}$ defined by

$$x_{n+1} = a'_n x_n + b'_n T x_n + c'_n v_n, n \geq 0 \tag{2.3.8}$$

is called the Mann iteration scheme with errors.

2.4 REVIEW OF SOME CONVERGENCE THEOREMS

Schu [36] introduced the following modified Mann iterative method. For any $x_1 \in C$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n \quad n \geq 1 \tag{2.4.1}$$

where $\{\alpha_n\}$ is a suitable sequence in the interval $(0, 1)$ and proved the following convergence theorem for asymptotically nonexpansive mapping T.

Theorem 2.4.1 *Let H be a Hilbert space, C a nonempty closed convex and bounded subset of H. Let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty), n \geq 1$ such that $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1 - \epsilon$ for all $n \geq 1$ and for some $\epsilon > 0$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n \quad n \geq 1,$$

converges strongly to some fixed point of T.

Rhoades [35] extended Theorem 1 to uniformly convex Banach spaces and proved convergence theorem for the modified Ishikawa type iteration processes.

Qihou [34] using the modified Mann iteration method introduced by Schu [36], proved the following convergence theorem

Theorem 2.4.2 *Let H be a Hilbert space, C a nonempty closed convex and bounded subset of H. Let $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian demicontractive mapping with $\{k_n\} \subset [1, \infty), n \geq 1$ such that $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $(0, 1)$ such that $0 < \epsilon < \alpha_n < 1 - k - \epsilon$ for all $n \geq 1$ and for some $\epsilon > 0$. Then for arbitrary $x_1 \in C$ define the sequence $\{x_n\}$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n \quad n \geq 1.$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some fixed point of T.

The modified Mann iteration method and the Ishikawa iteration method have been employed for the iterative approximation of asymptotically nonexpansive mappings and their generalizations.
Osilike and Aniagbosor [30] proved the following theorem in q-uniformly smooth Banach space.

Theorem 2.4.3 *Let $q \geq 1$ and let E be a real q-uniformly smooth Banach space. Let C a nonempty closed convex and bounded subset of E. Let $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian asymptotically demicontractive mapping with $\{k_n\} \subset [1, \infty)$, $n \geq 1$ satisfying $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$, and $\{c'_n\}$ be real sequences in $[0, 1]$ satisfying the conditions:*

i $\quad a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$

ii $\quad 0 < \epsilon \leq c_q(b')^{q-1}(1 + Lb_n)^q \leq \frac{1}{2}\left[q(1 - k)(1 + L)^{-(q-2)}\right] - \epsilon$, *for all $n \geq 1$ and for some $\epsilon > 0$;*

iii $\quad \sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} c_n < \infty \sum_{n=1}^{\infty} c'_n < \infty.$

Let $\{U_n\}, \{V_n\}$ be bounded sequences in C. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by

$$y_n = a_n x_n + b_n T^n x_n + c_n U_n \quad n \geq 1$$
$$x_{n+1} = a'_n x_n + b'_n T^n y_n + c'_n V_n \quad n \geq 1$$

converges strongly to some fixed point of T.

Igbokwe [15] reviewed and extended Osilike and Aniagbosor [30] result. In particular, he proved the following theorem.

Theorem 2.4.4 *Let X be a real Banach space and C a nonempty closed convex subset of X. Let $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian asymptotically demicontractive mapping with $\{k_n\} \subset [1, \infty), n \geq 1$ such that $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n\}, \{b_n\}, \text{and}\{c_n\}$ be real sequences in $[0, 1]$ satisfying the conditions:*

- i $\quad a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$
- ii $\quad \sum_{n=1}^{\infty} b'_n = \infty, \sum_{n=1}^{\infty}(b'_n)^2 < \infty$
- iii $\quad \sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c'_n < \infty$

Let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated from an arbitrary $x_1 \in C$ by

$$y_n = a_n x_n + b_n T^n x_n + c_n U_n \quad n \geq 1$$
$$x_{n+1} = a'_n x_n + b'_n T^n y_n + c'_n V_n \quad n \geq 1,$$

where $\{U_n\}, \{V_n\}$ are bounded sequences in C. Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Moore and Nnoli [24] proved a necessary and sufficient condition for approximating fixed point of asymptotically demicontractive mapping. More precisely, they proved the following theorem

Theorem 2.4.5 *Let X be a real Banach space and $T : X \to X$ be a uniformly L-Lipschitzian asymptotically demicontractive mapping with $\{k_n\} \subset [1, \infty)$ and $F(T) \neq \emptyset$. Let the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ be given by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n \quad n \geq 1.$$

Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n^2(k_n^2 - 1) < \infty$, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

In [14], Hu proved several strong convergence theorems for asymptotically demicontractive mappings using the Noor [25] iterative sequences with errors which improved the results of Igbokwe [15], Moore and Nnoli [24] and others. By using new analysis technique, Cho et al [10] proved the following theorem.

Theorem 2.4.6 *Let X be a real normed linear space, C be a nonempty closed convex subset of X and let $T : C \to C$ be a completely continuous and uniformly L-Lipschitzian asymptotically demicontractive mapping with sequence $\{k_n\} \subset [1, \infty), n \geq 1$ such that $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\},$ and $\{c'_n\}$ be real sequences in $[0, 1]$ satisfying the conditions:*

i $\quad a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$

ii $\quad \sum_{n=1}^{\infty} b'_n = \infty, \sum_{n=1}^{\infty}(b'_n)^2 < \infty$

iii $\quad \sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c'_n < \infty$

Let $\{U_n\}, \{V_n\}$ be bounded sequences in C. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by

$$y_n = a_n x_n + b_n T^n x_n + c_n U_n \quad n \geq 1$$

$$x_{n+1} = a'_n x_n + b'_n T^n y_n + c'_n V_n \quad n \geq 1$$

then sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

In 2011, T. Yuchao et al. [39] proved strong convergence theorems for a finite family of asymptotically demicontractive mapping in Banach spaces. Their results extended the corresponding ones announced by Osilike [27], Osilike and Aniagbosor [30], Igbokwe [15], Moore and Nnoli [24], Hu [14]. More precisely, they proved the following theorem:

Theorem 2.4.7 *Let X be a real Banach space and C a nonempty closed convex subset of X. Let $T_i : C \to C, i = 1, 2, 3, \cdots, r$ be a finite family of uniformly L_i-Lipschitzian asymptotically demicontractive mapping with $\{k_{in}\}_{n=1}^{\infty} \subset [1, \infty), n \geq 1$ such that $\sum_{n=1}^{\infty}(k_{in}^2 - 1) < \infty$. If $F := \bigcap_{i=n}^{r} F(T) \neq \emptyset$ and suppose the sequence $\{x_n\}_{n=1}^{\infty}$ is defined from an arbitrary $x_1 \in C$ by*

$$\begin{aligned} x_{n+1} &= a_{1n}x_n + b_{1n}T_1^n y_{1n} + c_{1n}U_{1n} \\ y_n &= a_{(i+1)}x_n + b_{(i+1)}T_{(i+1)}^{(i+1)} y_{(i+1)n} + c_{(i+1)}U_{(i+1)n} \quad n \geq 1, \end{aligned}$$

where $\{a_{in}\}, \{b_{in}\}, \{c_{in}\}$ are real sequences in $[0, 1]$ satisfying $a_{in} + b_{in} + c_{in} = 1$ for any $i = 1, 2, 3, \cdots, r$ and $n \geq 1, \{U_{in}\}$, are bounded sequences in C, $i = 1, 2, 3, \cdots, r-1$, $y_{rn} = x_n$ and satisfying the conditions

i $\quad \sum_{n=1}^{\infty} b_{1n} = \infty, \sum_{n=1}^{\infty} b_{2n} < \infty$

ii $\quad \sum_{n=1}^{\infty}(b_{1n}))^2 < \infty, \sum_{n=1}^{\infty} c_{in} < \infty, i = 1, 2, 3, \cdots, r$

then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{r}$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$

Inspired and motivated by these results, the purpose of this project report therefore as we have noted earlier is to propose an algorithm (analogous to (1.1.2)) for asymptotically demicontractive mappings which will yield strong convergence; and in particular show that the proposed algorithm converges strongly to the least norm element of the set of fixed points of asymptotically demicontractive mappings.

Chapter 3

MODIFIED KRASNOSELSKII-MANN ALGORITHM FOR A CLASS OF NONLINEAR MAPPING

In this chapter, we investigated a modified Krasnoselkii-Mann iteration, which is intended to converge strongly. The modified Krasnoselskii-Mann algorithm we want to investigate is the following:

$$\begin{aligned} v_n &= (1-t_n)x_n \\ x_{n+1} &= (1-\alpha_n)v_n + \alpha_n T^n v_n \quad n \geq 1 \end{aligned} \tag{3.0.1}$$

where $\{t_n\}_{n=1}^{\infty} \subset (0,1), \{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, are real sequences such that $\lim\limits_{n\to\infty} t_n = 0$ and $\sum\limits_{n=1}^{\infty} t_n = \infty$

Lemma 3.1

Let $\{\alpha_n\} \in [0,1)$, for all $n \geq 1$. Then $\prod_{n=1}^{\infty}(1+\alpha_n) < \infty$ if and only if $\sum_{n=1}^{\infty} \alpha_n < \infty$

Proof of Lemma (3.1):

($\Rightarrow$) Let $\sum_{n=1}^{\infty} \alpha_n < \infty$. Then for any $n \geq 1$ we have $(1+\alpha_n) \leq e^{\alpha_n}$ Hence,

$$\prod_{n=1}^{\infty}(1+\alpha_n) \leq \prod_{n=1}^{\infty} e^{\alpha_n} = e^{\sum_{n=1}^{\infty}\alpha_n} < \infty.$$

($\Leftarrow$) Let $\prod_{n=1}^{\infty}(1+\alpha_n) < \infty$. Then

$$\sum_{j=1}^{n} \alpha_j \leq \prod_{j=1}^{n}(1+\alpha_j) \ \forall\, n \geq 1.$$

This follows by induction since the inequality clearly holds for $n = 1$ and if we assume that it holds for n, the

$$\begin{aligned}
\sum_{j=1}^{n+1} \alpha_j &= \sum_{j=1}^{n} \alpha_j + \alpha_{n+1} \\
&\leq \prod_{j=1}^{n}(1+\alpha_j) + \alpha_{n+1} \\
&\leq \prod_{j=1}^{n}(1+\alpha_j) + \alpha_{n+1}\prod_{j=1}^{n}(1+\alpha_j) = (1+\alpha_{n+1})\prod_{j=1}^{n}(1+\alpha_j) = \prod_{j=1}^{n+1}(1+\alpha_j).
\end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} \alpha_n \leq \prod_{n=1}^{\infty}(1+\alpha_n) < \infty.$$

3.1 MAIN RESULTS

We start with the following which will play crucial role in the proof of our main theorem:

Lemma 3.2 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\gamma_n + \sigma_n, \ n \geq 1, \tag{3.1.1}$$

where $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence of real numbers which is bounded above, $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ satisfies $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \gamma_n$$

holds.

Proof. Since $\{\gamma_n\}$ is bounded above, then for arbitrary but fixed $m \in N$, we define

$$\lambda_m = \sup_{j\geq m} \gamma_j.$$

Then for $n \geq m$, we obtain

$$\alpha_{n+1} - \alpha_n + \alpha_n(a_n - \lambda_m) \leq \sigma_n.$$

Let

$$b_n = a_n - \lambda_m. \tag{3.1.2}$$

Then

$$\begin{aligned} b_{n+1} &= a_{n+1} - \lambda_m \\ &\leq (1-\alpha_n)a_n + \alpha_n\gamma_n + \sigma_n - \lambda_m \\ &= a_n - \lambda_m - \alpha_n a_n + \alpha_n\gamma_n + \sigma_n \\ &\leq a_n - \lambda_m - \alpha_n a_n + \alpha_n\lambda_m + \sigma_n \\ &= a_n - \lambda_m - \alpha_n(a_n - \lambda_m) + \sigma_n \\ &= (1-\alpha_n)(a_n - \lambda_m) + \sigma_n \\ &= (1-\alpha_n)b_n + \sigma_n. \end{aligned}$$

This implies that $\{b_n\}$ is bounded and so $\{a_n\}$ is also bounded. Moreover,

$$b_{n+1} \leq \Big[\prod_{j=m}^{n}(1-\alpha_j)\Big]\lambda_m + \sum_{j=m}^{n}\sigma_j. \tag{3.1.3}$$

Since $\sum_{n=1}^{\infty}\alpha_n = \infty$, we have that $\lim\limits_{n\to\infty}\prod_{j=m}^{n}(1-\alpha_j) = 0$. Thus, it follows from (3.1.3) that

$$\limsup_{n\to\infty} b_n = \limsup_{n\to\infty} b_{n+1} \leq 0,$$

which is equivalent to

$$\limsup_{n\to\infty} a_n \leq \lambda_m,$$

and letting $m \to \infty$, we obtain

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \gamma_n.$$

Lemma 3.3:

Let C be a closed, convex nonempty subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotically demicontractive map with contractive constant $k \in [0, 1)$ and sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$. Let $(I - T)$ be demiclosed at zero. Then the fixed point set F(T) (assumed to be nonempty) of T is closed and convex.

Proof of Lemma 3.3:

Let $x, y \in F(T), \lambda \in [0, 1]$, and set $p = \lambda x+(1-\lambda)y$. Observe that $\|x-p\| = (1-\lambda)\|x-y\|$, and $\|y - p\| = \lambda\|x - y\|$.

Now, using (2.2.12), we obtain that

$$\begin{aligned}
\|T^np - p\|^2 &= \|\lambda(x - T^np) + (1 - \lambda)(y - T^np)\|^2 \\
&= \lambda\|x - T^np\|^2 + (1 - \lambda)\|y - T^np\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \\
&\leq \lambda(k_n^2\|x - p\|^2 + k\|T^np - p\|^2) \\
&\quad + (1 - \lambda)(k_n^2\|y - p\|^2 + k\|T^np - p\|^2) - \lambda(1 - \lambda)\|x - y\|^2 \\
&= \lambda(1 - \lambda)^2k_n^2\|x - y\|^2 + \lambda^2k_n^2\|x - y\|^2 - k_n^2\lambda^3\|x - y\| \\
&\quad + k\|T^np - p\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \\
&= \lambda(1 - \lambda)\left[k_n^2(1 - \lambda) + \lambda k_n^2 - 1\right]\|x - y\|^2 + k\|T^np - p\|^2 \\
&= \lambda(1 - \lambda)(k_n^2 - 1)\|x - y\|^2 + k\|T^np - p\|^2 \\
\text{i.e.,}\quad \|T^np - p\|^2 &\leq \frac{\lambda(1 - \lambda)}{1 - k}(k_n^2 - 1)\|x - y\|^2
\end{aligned}$$

taking limit of both sides as n tends to infinity and applying sandwich theorem, we obbtain that $\|T^np - p\| \to 0$ as n $\to \infty$

Let $x_n := T^np,\ n \geq 1$. Then $x_n \to p$ as $n \to \infty$; $x_n - Tx_n \to 0$ as $n \to \infty$ and by demiclosedness property of $(I - T)$ at zero, we have that $(I - T)p = 0$. Thus $Tp = p$, which implies that $p \in F(T)$. That is $F(T)$ is convex.

Next we show that $F(T)$ is closed. Let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary sequence in $F(T)$ such that $x_n \to x \in H$. We show that $x \in F(T)$

Now since $\{x_n\} \subseteq F(T)$, then $Tx_n = x_n$ for all $n \geq 1$. Since $Tx_n = x_n$, and $x_n \to x$ then $Tx_n \to x$, and hence $x_n - Tx_n \to 0$ and the demiclosedness property of $I - T$ at 0 then yields $Tx = x$. Thus, $x \in F(T)$.

We now prove the following theorem.

Theorem 3.1.1 *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be a uniformly L-Lipschitzian asymptotically demicontractive mapping with $F(T) \neq \emptyset$, $(I - T)$ demiclosed at zero, and sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. For arbitrary $x_1 \in C$ define the sequence $\{x_n\}_{n=1}^{\infty}$ by*

$$\begin{aligned} v_n &= P_C[(1 - t_n)x_n] \\ x_{n+1} &= (1 - \alpha_n)v_n + \alpha_n T^n v_n \quad n \geq 1 \end{aligned} \tag{3.1.3}$$

where $\{t_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty}$ are real sequences satisfying the following conditions given below:

i $\lim_{n\to\infty} t_n = 0, \sum_{n=1}^{\infty} t_n = \infty$

ii $\{\alpha_n\}_{n=1}^{\infty} \subset (0, b) \subset (0, 1)$ *with* $0 < b < 1 - k,$ where k is the constant appearing in (2.2.18)

iii $\frac{\mathrm{t_n}}{\alpha_\mathrm{n}} \to 0$ as $\mathrm{n} \to \infty$

Then the generated sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the least norm element of the fixed point set of T. **Proof:** Let $p \in F(T)$. Then

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||v_n - \alpha_n v_n + \alpha_n T^n v_n - p||^2 \\ &= ||v_n - p - \alpha_n(v_n - T^n v_n)||^2 \\ &= ||v_n - p||^2 + \alpha_n^2 ||v_n - T^n v_n||^2 \\ &\quad -2\alpha_n \langle v_n - T^n v_n, v_n - p\rangle \end{aligned} \tag{3.1.4}$$

but $2\langle v_n - T^n v_n, v_n - p\rangle \geq (1-k)||v_n - T^n v_n||^2 - (k_n^2 - 1)||v_n - p||^2$ by (2.2.19). Using it in (3.2.2) yields:

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq ||v_n - p||^2 + \alpha_n^2 ||v_n - T^n v_n||^2 \\
&\quad -\alpha_n(1-k)||v_n - T^n v_n||^2 + \alpha_n(k_n^2 - 1)||v_n - p||^2 \\
&= [1 + \alpha_n(k_n^2 - 1)]||v_n - p||^2 - \alpha_n(1 - k - \alpha_n)||v_n - T^n v_n||^2 \\
\text{But } [1 + \alpha_n(k_n^2 - 1)] &\leq (1 + k_n^2 - 1) = k_n^2, \quad (\text{since } \{\alpha_\text{n}\} < 1\ \forall\ \text{n} \in \mathbb{N}.)
\end{aligned}$$

So,

$$||x_{n+1} - p||^2 \leq k_n^2 ||v_n - p||^2 - \alpha_n(1 - k - \alpha_n)||v_n - T^n v_n||^2 \tag{3.1.5}$$

Again from (3.2.1):

$$\begin{aligned}
x_{n+1} - v_n &= -\alpha_n(v_n - T^n v_n) \\
\text{or } \frac{1}{\alpha_n}(v_n - x_{n+1}) &= (v_n - T^n v_n)
\end{aligned} \tag{3.1.6}$$

Using (3.2.4) in (3.2.3), we have

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq k_n^2 ||v_n - p||^2 - \alpha_n(1 - k - \alpha_n)\frac{1}{\alpha_n^2}||v_n - x_{n+1}||^2 \\
&= k_n^2 ||v_n - p||^2 - (1 - k - \alpha_n)\frac{1}{\alpha_n}||v_n - x_{n+1}||^2 \\
i.e., \quad ||x_{n+1} - p||^2 &\leq k_n^2 ||v_n - p||^2 - (1 - k - \alpha_n)\frac{1}{\alpha_n}||v_n - x_{n+1}||^2 \qquad (3.1.7)
\end{aligned}$$

since $\{\alpha_n\} \subset (0, b)$ $(0 < \alpha_n \leq b < 1 - k)$, we have that $(1 - k - \alpha_n)\frac{1}{\alpha_n} \geq (1 - k - b)\frac{1}{\alpha_n} > 0$ and so (3.2.5) gives

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq k_n^2 ||v_n - p||^2 \\
||x_{n+1} - p|| &\leq k_n ||v_n - p||
\end{aligned} \tag{3.1.8}$$

Now still from (3.2.1):

$$
\begin{aligned}
v_n &= P_C[(1-t_n)x_n] \\
||v_n - p|| &= ||P_C[(1-t_n)x_n] - p|| \\
&= ||P_C[(1-t_n)x_n] - P_C(p)|| \\
&\leq ||(1-t_n)x_n - p|| \\
&= ||(1-t_n)x_n - t_n p + t_n p - p|| \\
&= ||(1-t_n)(x_n - p) - t_n p|| \\
&\leq (1-t_n)||x_n - p|| + t_n||p|| \\
\text{i.e.,} \quad ||v_n - p|| &\leq (1-t_n)||x_n - p|| + t_n||p|| \qquad (3.1.9)
\end{aligned}
$$

Using (3.2.7) in (3.2.6), we obtain that:

$$
\begin{aligned}
||x_{n+1} - p|| &\leq k_n[(1-t_n)||x_n - p|| + t_n||p||] \\
&= k_n(1-t_n)||x_n - p|| + k_n t_n||p|| \\
\text{i.e.,} \quad ||x_{n+1} - p|| &\leq k_n(1-t_n)||x_n - p|| + k_n t_n||p|| \\
&\leq k_n(1-t_n)\max\{||x_n - p||, ||p||\} + k_n t_n \max\{||x_n - p||, ||p||\} \\
&= k_n \max\{||x_n - p||, ||p||\} \\
\text{i.e.,} \quad ||x_{n+1} - p|| &\leq k_n \max\{||x_n - p||, ||p||\} \\
\text{but} \quad ||x_n - p|| &\leq k_{n-1}\max\{||x_{n-1} - p||, ||p||\} \\
\text{and} \quad ||x_{n-1} - p|| &\leq k_{n-2}\max\{||x_{n-2} - p||, ||p||\} \\
\text{i.e.,} \quad ||x_n - p|| &\leq k_n k_{n-1} k_{n-2}\max\{||x_{n-2} - p||, ||p||\} \\
&\vdots \\
&\leq \prod_{i=1}^{n} k_i \max\{||x_0 - p||, ||p||\} < \infty,
\end{aligned}
$$

since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, implies by Lemma 3.1 that $\prod_{n=1}^{\infty} k_n < \infty$. This implies that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Again,

$$\begin{aligned}
||v_n - p||^2 &= ||P_C[(1-t_n)(x_n)] - P_C(p)||^2 \\
&\leq ||(1-t_n)(x_n - p) - t_n p||^2 \\
&= (1-t_n)^2||x_n - p||^2 + t_n^2||p||^2 - 2t_n(1-t_n)\left\langle x_n - p, p\right\rangle \\
i.e, \quad ||v_n - p||^2 &\leq (1-t_n)^2||x_n - p||^2 + t_n^2||p||^2 \\
&\quad + 2t_n(1-t_n)\left\langle p - x_n, p\right\rangle. \qquad (3.1.10)
\end{aligned}$$

Using (3.2.8) in (3.2.5) gives

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq k_n^2\Big[(1-t_n)^2||x_n - p||^2 + t_n^2||p||^2 \\
&\quad +2t_n(1-t_n)\left\langle p - x_n, p\right\rangle\Big] - (1-k-\alpha_n)\frac{1}{\alpha_n}||v_n - x_{n+1}||^2 \\
&= (1-t_n)^2||x_n - p||^2 + (k_n^2 - 1)(1-t_n)^2||x_n - p||^2 + k_n^2 t_n^2||p||^2 \\
&+ 2k_n^2 t_n(1-t_n)\langle p - x_n, p\rangle - (1-k-\alpha_n)\frac{1}{\alpha_n}||v_n - x_{n+1}||^2 \\
&= (1-t_n)^2||x_n - p||^2 - t_n\Big[-2k_n^2(1-t_n)\langle p - x_n, p\rangle \\
&\quad -k_n^2 t_n||p||^2 + (1-k-\alpha_n)\frac{1}{\alpha_n t_n}||v_n - x_{n+1}||^2\Big] \\
&\quad +(k_n^2 - 1)(1-t_n)^2||x_n - p||^2 \\
i.e., \quad ||x_{n+1} - p||^2 &\leq (1-t_n)^2||x_n - p||^2 - t_n Y_n + \sigma_n \qquad (3.1.11)
\end{aligned}$$

where

$$\begin{aligned}
Y_n &= -2k_n^2(1-t_n)\left\langle p - x_n, p\right\rangle - k_n^2 t_n||p||^2 \\
&\quad +(1-k-\alpha_n)\frac{1}{\alpha_n t_n}||v_n - x_{n+1}||^2, \qquad (3.1.12)
\end{aligned}$$

and $\sigma_n = +(k_n^2 - 1)(1-t_n)^2||x_n - p||^2$. Since $\{(1-k-\alpha_n)\frac{1}{\alpha_n t_n}||v_n - x_{n+1}||^2\}$ is bounded below, and $\{-2k_n^2(1-t_n)\langle p - x_n, p\rangle\}$ and $\{k_n^2 t_n||p||^2\}$ are bounded, then Y_n is bounded below. So, using (3.2.9) and the condition $\sum_{n=1}^{\infty} t_n = \infty$, it follows from Lemma 3.2 that

$$\begin{aligned}
\limsup_{n\to\infty} ||x_n - p||^2 &\leq \limsup_{n\to\infty}(-Y_n) \\
&= -\liminf_{n\to\infty} Y_n. \qquad (3.1.13)
\end{aligned}$$

$$\begin{aligned}
\text{This implies that } \liminf_{n\to\infty} Y_n &\leq -\limsup_{n\to\infty} ||x_n - p||^2 < \infty \\
\text{(since } \{||x_n - p||^2\} \text{ is bounded).} & \qquad (3.1.14)
\end{aligned}$$

But

$$\begin{aligned}
\liminf_{n\to\infty} Y_n &= \liminf_{n\to\infty}[-2\langle p - x_n, p\rangle \\
&\quad + (1 - k - \alpha_n)\frac{1}{\alpha_n t_n}\|v_n - x_{n+1}\|^2] \qquad (3.1.15)
\end{aligned}$$

Thus, by the property of liminf, there exists a subsequence $\{Y_{n_j}\}_{j=1}^{\infty}$ of $\{Y_n\}_{n=1}^{\infty}$ such that

$$\liminf_{n\to\infty} Y_n = \liminf_{n\to\infty} Y_{n_j}.$$

This implies that

$$\liminf_{n\to\infty} Y_n = \lim_{j\to\infty}\left[-2\left\langle p - x_{n_j}, p\right\rangle + \frac{(1 - k - \alpha_{n_j})}{\alpha_{n_j}}\frac{1}{t_{n_j}}||v_{n_j} - x_{n_j+1}||^2\right]. \quad (3.1.16)$$

Since $\liminf_{n\to\infty} Y_n < \infty$, it follows that $\lim_{j\to\infty}\left[-2\left\langle p - x_{n_j}, p\right\rangle + \frac{(1-k-\alpha_{n_j})}{\alpha_{n_j}}\frac{1}{t_{n_j}}||v_{n_j} - x_{n_j+1}||^2\right] < \infty$. Thus, since $\{x_n\}$ is bounded we have that

$$\left\{\frac{(1 - k - \alpha_{n_j})}{\alpha_{n_j}}\frac{1}{t_{n_j}}||v_{n_j} - x_{n_j+1}||^2\right\}_{n=1}^{\infty}$$

is bounded. Furthermore, from condition (ii) we have $-\alpha_n > -b > -(1 - k)$ and hence

$$1 - k - \alpha_n > 1 - k - b > 0.$$

Thus

$$\frac{(1 - k - \alpha_n)}{\alpha_n t_n}||v_n - x_{n+1}||^2 > \frac{(1 - k - b)}{\alpha_n t_n}||v_n - x_{n+1}||^2,$$

and hence

$$\frac{1}{\alpha_{n_j} t_{n_j}}||v_{n_j} - x_{n_j+1}||^2 < \frac{(1 - k - \alpha_{n_j})}{(1 - k - b)\alpha_{n_j} t_{n_j}}||v_{n_j} - x_{n_j+1}||^2.$$

It follows that $\{\frac{1}{\alpha_{n_j} t_{n_j}}||v_{n_j} - x_{n_j+1}||^2\}$ is bounded.

From (3.2.4) we have

$$\begin{aligned}
||v_{n_j} - T^{n_j} v_{n_j}||^2 &= \frac{1}{\alpha_{n_j}^2}||v_{n_j} - x_{n_j+1}||^2 \\
&= \frac{t_{n_j}}{\alpha_{n_j}} \frac{||v_{n_j} - x_{n_j+1}||^2}{\alpha_{n_j} t_{n_j}} \to 0 \text{ as} j \to \infty,
\end{aligned}$$

since $\frac{t_{n_j}}{\alpha_{n_j}} \to 0$ as$j \to \infty$, and $\{\frac{1}{\alpha_{n_j} t_{n_j}}||v_{n_j} - x_{n_j+1}||^2\}$ is bounded. Observe that

$$\begin{aligned}
||x_{n+1} - x_n|| &= ||(1-\alpha_n)v_n + \alpha_n T^n v_n - x_n|| \\
&= ||v_n - x_n - \alpha_n(v_n - T^n v_n)|| \\
&\leq t_n||x_n|| + \alpha_n||v_n - T^n v_n||.
\end{aligned}$$

Thus,

$$||x_{n_j+1} - x_{n_j}|| \leq t_{n_j}||x_{n_j}|| + \alpha_{n_j}||v_{n_j} - T^{n_j} v_{n_j}|| \to 0 \text{ as } j \to \infty.$$

Also

$$\begin{aligned}
||v_{n_j} - x_{n_j+1}|| &\leq ||v_{n_j} - x_{n_j}|| + ||x_{n_j} - x_{n_j+1}|| \\
&\leq t_{n_j}||x_{n_j}|| + ||x_{n_j} - x_{n_j+1}|| \to 0 \text{ as } j \to \infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|v_{n_j-1} - T^{n_j-1} v_{n_j}\|^2 &= \|v_{n_j-1} - T^{n_j-1} v_{n_j} + T^{n_j} v_{n_j} - T^{n_j} v_{n_j}\|^2 \\
&\leq \|v_{n_j-1} - T^{n_j-1} v_{n_j}\|^2 + L\|v_{n_j-1} - v_{n_j}\|^2, \ (\mathrm{L} = \mathrm{T}^{\mathrm{n_j}-1})
\end{aligned}$$

and

$$\begin{aligned}
\|v_{n_j-1} - v_{n_j}\| &\leq \|(1-t_{n_j-1})x_{n_j-1} - (1-t_{n_j-1})x_{n_j}\| \\
&\leq \|x_{n_j-1} - x_{n_j}\| + \|t_{n_j-1}x_{n_j-1} - t_{n_j}x_{n_j-1}\| \\
&= \|x_{n_j-1} - x_{n_j}\| + \|t_{n_j-1}x_{n_j-1} - t_{n_j-1}x_{n_j-1} + t_{n_j-1}x_{n_j} - t_{n_j}x_{n_j}\| \\
&\leq \|x_{n_j-1} - x_{n_j}\| + t_{n_j-1}\|x_{n_j-1} - x_{n_j}\| + |t_{n_j-1} - t_{n_j}|\|x_{n_j}\| \\
&\to 0 \text{ as } j \to \infty.
\end{aligned}$$

Thus,

$$||v_{n_j-1} - T^{n_j-1}v_{n_j}||^2 \to 0 \text{ as } n \to \infty.$$

Hence,

$$\begin{aligned}
||v_{n_j} - Tv_{n_j}|| &\leq ||v_{n_j} - T^{n_j}v_{n_j}|| + ||T^{n_j}v_{n_j} - Tv_{n_j}|| \\
&\leq ||v_{n_j} - T^{n_j}v_{n_j}|| + L||T^{n_j-1}v_{n_j} - v_{n_j}|| \\
&\leq ||v_{n_j} - T^{n_j}v_{n_j}|| + L||T^{n_j-1}v_{n_j} - T^{n_j-1}v_{n_j-1}|| + L||T^{n_j-1}v_{n_j-1} - v_{n_j}|| \\
&\leq ||v_{n_j} - T^{n_j}v_{n_j}|| + L||T^{n_j-1}v_{n_j-1} - v_{n_j-1}|| \\
&\quad +L(1+L)||v_{n_j} - v_{n_j-1}|| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Since $(I - T)$ is demiclosed at 0, it follows that any weak cluster point of $\{v_{n_j}\}$ is a fixed point of T. Furthermore, since $F(T)$ is closed and convex, $P_{F(T)} : H \to F(T)$ is well defined. Thus, for $p = P_{F(T)}(0)$, we obtain from (3.2.14) and Lemma (1.5.4) (i) that

$$\begin{aligned}
\liminf_{n\to\infty} Y_n &= -2\liminf_{j\to\infty}[\langle x_{n_j} - P_{F(T)}(0), P_{F(T)}(0)\rangle] \leq 0. \\
-\liminf_{n\to\infty} Y_n &= 2\liminf_{j\to\infty}[\langle x_{n_j} - P_{F(T)}(0), P_{F(T)}(0)\rangle] \geq 0.
\end{aligned}$$

Consequently, from (3.2.11) we have that

$$0 \leq \limsup_{n\to\infty} ||x_n - P_{F(T)}(0)||^2 \leq -\liminf_{n\to\infty} Y_n \leq 0$$

$$\text{i.e.,} \quad \limsup_{n\to\infty} ||x_n - P_{F(T)}(0)|| = 0$$

Clearly $\|P_{F(T)}(0)\| = \|p\| \leq \|y\| \;\forall\, y \in F(T)$

Thus, the generated sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the least norm element of fixed point set of T.

Theorem 3.1.2 *Let H be a real Hilbert space. Let $T : C \to C$ be an asymptotically k-strictly Pseudocontractive mapping with $F(T) \neq \emptyset$, $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1, \sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. For arbitrary $x_1 \in C$ define the sequence $\{x_n\}_{n=1}^{\infty}$ by*

$$\begin{aligned}
v_n &= P_C[(1-t_n)x_n] \\
x_{n+1} &= (1-\alpha_n)v_n + \alpha_n T^n v_n \quad n \geq 1 \qquad (3.1.17)
\end{aligned}$$

where $\{t_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty}$ *are real sequences satisfying the following conditions given below:*

i $\quad \lim_{n\to\infty} t_n = 0, \sum_{n=1}^{\infty} t_n = \infty$

ii $\quad \{\alpha_n\}_{n=1}^{\infty} \subset (0,b) \subset (0,1)$ *with* $0 < b < 1-k,$ where k is the constant appearing in (2.2.18)

iii $\quad \frac{t_n}{\alpha_n} \to 0$ *as* $n \to \infty$

Then the generated sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the least norm element of the fixed point set of T.

Theorem 3.1.3 *Let H be a real Hilbert space. Let* $T : C \to C$ *be an asymptotically nonexpansive mapping with* $F(T) \neq \emptyset$, $\{k_n\} \subset [1,\infty)$ *with* $\lim_{n\to\infty} k_n = 1, \sum_{n=1}^{\infty}(k_n^2 - 1) < \infty.$ *For arbitrary* $x_1 \in C$ *define the sequence* $\{x_n\}_{n=1}^{\infty}$ *by*

$$\begin{aligned} v_n &= P_C[(1-t_n)x_n] \\ x_{n+1} &= (1-\alpha_n)v_n + \alpha_n T^n v_n \quad n \geq 1 \end{aligned} \tag{3.1.18}$$

where $\{t_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty}$ *are real sequences satisfying the following conditions given below:*

i $\quad \lim_{n\to\infty} t_n = 0, \sum_{n=1}^{\infty} t_n = \infty$

ii $\quad \{\alpha_n\}_{n=1}^{\infty} \subset (0,b) \subset (0,1)$ *with* $0 < b < 1-k,$ where k is the constant appearing in (2.2.18)

iii $\quad \frac{t_n}{\alpha_n} \to 0$ *as* $n \to \infty$

Then the generated sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the least norm element of the fixed point set of T.

Prototype for the sequences $\{t_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ are respectively given by

$$\begin{aligned} t_n &= \frac{1}{n+1} \\ \alpha_n &= \sqrt{\frac{1}{n+1}} \end{aligned}$$

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