# WEAK AND STRONG CONVERGENCE OF AN ITERATIVE ALGORITHM FOR LIPSCHITZ PSEUDO-CONTRACTIVE MAPS IN HILBERT SPACES

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# Weak and Strong Convergence of an Iterative Algorithm for Lipschitz Pseudo-contractive Maps in Hilbert Spaces

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# CERTIFICATION

Ingbianfam, Ezekiel, Veluhan, a postgraduate student in the Department of Mathematics with registration number PG/M.Sc./10/57416 has satisfactorily completed the requirements for research work for the degree Master of Science in Mathematics. The work embodied in this project report is original and has not been submitted in part or full for any other diploma or degree of this or any other university.

| | |
|---|---|
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# DEDICATION

This work is gratefully dedicated to my father Mr. Mende Simon I.

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# Contents

# ABSTRACT

Let $H$ be a real Hilbert space and $K$ a nonempty, closed convex subset of $H$.Let $T : K \to K$ be Lipschitz pseudo-contractive map with a nonempty fixed points set. We introduce a modified Ishikawa iterative algorithm for Lipschitz pseudo-contractive maps and prove that our new iterative algorithm converges strongly to a fixed point of $T$ in real Hilbert space.

# Chapter 1

# Introduction

## 1.1 General Introduction

The contribution of this thesis falls under a branch of mathematics called Functional Analysis. Functional Analysis as an independent mathematical discipline started at the turn of the 19th century and was finally established in 1920's and 1930's, on one hand under the influence of the study of specific classes of linear operators-integral operators and integral equations connected with them-and on the other hand under the influence of the purely intrinsic development of modern mathematics with its desire to generalize and thus to clarify the true nature of some regular behaviour. Quantum Mechanics also had a great influence on the development of Functional Analysis, since its basic concepts, for example energy, turned out to be linear operators (which physicists at first rather loosely interpreted as infinite dimensional matrices) on infinite dimensional spaces. In the early stages of the development of Functional Analysis the problems studied were those that could be stated and solved in terms of linear operators on elements of the space alone. But as the concept of a space was being developed and deepened, the concept of a function was being developed and generalized. In the end, it became necessary to consider mapping (not necessary linear) from one space into another. One of the central problems in non-linear Functional Analysis is the study of such mappings. In the modern view, Functional Analysis is seen as the study of complete normed vector spaces over the real or complex numbers. Such studies are narrowed to the study of Banach spaces. An important example is a Hilbert space, where the norm arises from an inner product.

This project sets to solve the problem of constructing an iterative scheme for approximating fixed points of Lipschitz Pseudo-contractive Maps in Hilbert spaces. We introduced a modified Ishikawa iterative algorithm and prove that if
$F(T) = \{x \in H : Tx = x\} \neq \emptyset$, then our proposed iterative algorithm converges strongly to a fixed point of $T$. No compactness assumption is imposed on $T$ and no further requirement is imposed on $F(T)$.

We proceed with the definitions of some basic terms, and the introduction of various non linear operators studied in this project.

**Definition 1.1** : Let $K$ be a non empty subset of a real normed space $E$ and let $T : K \to K$ be a map. A point $x \in K$ is said to be a fixed point of $T$ if $Tx = x$. We shall denote the set of fixed points of $T$ by $F(T)$.

**Definition 1.2 (Convex Set)** : The set $C$ of a real vector space $X$ is called convex if, for any pair of points $x, y \in C$, the closed segment with extremities $x, y \in C$ that is, the set $\{\lambda x + (1-\lambda)y : \lambda \in [0, 1]\}$ is contained in $C$. A subset $C$ of a real normed space is called bounded if there exists $M > 0$ such that $\|x\| \leq M$ $\forall x \in C$.

**Definition 1.3** : Let $K$ be a non-empty closed convex subset of a Hilbert space $H$. The (metric or nearest point) projection onto $K$ is the mapping $P_k : H \to K$ which assigns to each $x \in H$ the unique point $P_k x$ in $K$ with the property
$\|x - P_k x\| = min\{\|x - y\| : y \in K\}$.

**Lemma 1.1**: Given $x \in H$ and $z \in K$. Then $z = P_k x$ if and only if
$\langle x - z, y - z\rangle \leq 0$ for all $y \in K$.

As a consequence we have that

(i) $\|P_k x - P_k y\|^2 \leq \langle x - y, P_k x - P_k y\rangle$ for all $x, y \in H$; that is, the projection is non expansive;

(ii) $\|x - P_k x\|^2 \leq \|x - y\|^2 - \|y - P_k x\|^2 \quad \forall x \in H$ and $y \in K$

(iii) If $K$ is a closed subspace, then $P_k$ coincides with the orthogonal projection from $H$ onto $K$; that is, for $x \in H, x - P_k x$ is orthogonal to $K$ (i.e. $\langle x - P_k x, y\rangle = 0$ for $y \in K$).

If $K$ is a closed convex subset with a particularly simple structure, then the projection $P_k$ has a closed form expression as described below:

(a.) If $K = \{x \in H : \|x - u\| \leq r\}$ is a closed ball centred at $u \in H$ with radius $r > 0$, then

$$P_k x = \begin{cases} u + r\frac{(x-u)}{\|x-u\|}, if x \notin K \\ x, if x \in K. \end{cases}$$

(b.) If $K = [a, b]$ is a closed rectangle in $\Re^n$, where $a = (a_1, a_2, ..., a_n)^T$ and $b = (b_1, b_2, ..., b_n)^T$ where $T$ is the transpose, then, for $1 \leq i \leq n, P_k x$ has the $i^{th}$ coordinate given by

$$(P_k x)i = \begin{cases} a_i, & if \quad x_i < a_i, \\ x_i, & if \quad x_i \in [a_i, b_i], \\ b_i, & if \quad x_i > b_i. \end{cases}$$

(c.) If $K = \{y \in H : \langle a, y\rangle = \alpha\}$ is a hyperplane, with $a \neq 0$ and $\alpha \in \Re$, then $P_k x = x - \frac{\langle a,x\rangle - \alpha}{\|a\|^2} a.$

(d.) If $K = \{y \in H : \langle a, y\rangle \leq \alpha\}$ is a closed half space, with $a \neq o$ and $\alpha \in \Re$, then

$$P_k x = \begin{cases} x - \frac{\langle a,x\rangle - \alpha}{\|a\|^2} a, & if \quad \langle a, x\rangle > \alpha \\ x, \quad if \quad \langle a, x\rangle \leq \alpha. \end{cases}$$

(e) If $K$ is the range of an $m \times n$ matrix $A$ with full column rank, then

$P_k x = A(A^* A)^{-1} A^* x$

where $A^*$ is the adjoint of $A$.

## 1.2 Demiclosedness Principles

A fundamental result in the theory of nonexpansive mappings is Browder's demiclosedness principle.

**Definition 1.2.1** : A mapping $T : K \to H$ is said to be demiclosed (at $y$) if the conditions that $\{x_n\}$ converges weakly to $x$ and that $\{Tx_n\}$ converges strongly to $y$ imply that $x \in K$

and $Tx = y$. Moreover, we say that $H$ satisfies the demiclosedness principle if for any closed convex subset $K$ of $H$ and any nonexpansive mapping $T : K \to H$, the mapping $I - T$ is demiclosed.

The demiclosedness principle plays an important role in the theory of non expansive mappings (and other classes of non linear mappings as well). In 1965, Browder [9] gave the following demiclosed principle for non expansive mappings in Hilbert spaces.

**Theorem 1.1(Browder [9])** Let $K$ be a non empty closed convex subset of a real Hilbert space $H$. Let $T$ be a non expansive mapping on $K$ into itself, and let $\{x_n\}$ be a sequence in $K$. If $x_n \rightharpoonup w$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tw = w$.

## 1.3 Nonlinear Mappings

The following definitions contains the nonlinear mappings we are working with and that will appear throughout the entire chapters.

**Definition 1.3.1** Let $K$ be a nonempty subset of a real Hilbert space $H$. The mapping $T : K \to H$ is called

(i) Lipschitz or Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall \quad x, y \in K \tag{1.1}$$

If $L = 1$, then $T$ is called nonexpansive; and if $L < 1$, then $T$ is called a contraction. It is easy to see from (1.1) that every contraction mapping is nonexpansive and every nonexpansive mapping is Lipschitz.

**Definition 1.3.2**: Let $K$ be a non empty subset of a real Hilbert space $H$.

(i) A mapping $T : K \to H$ is called pseudo-contractive if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 \quad \forall \quad x, y \in K \tag{1.2}$$

(ii) A mapping $T : K \to H$ is called a strict pseudo-contraction if for all $x, y \in K$ there exists a constant $\lambda \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \lambda\|(I - T)x - (I - T)y\|^2, \quad \forall \quad x, y \in K \tag{1.3}$$

Inequality (1.2) can be equivalently written as:

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \|(I-T)x - (I-T)y\|^2 \forall x, y \in K \tag{1.4}$$

i.e. set $A = I - T$

$$\begin{aligned}
\Rightarrow \|(I-A)x - (I-A)y\|^2 &\leq \|x-y\|^2 + \|Ax - Ay\|^2 \\
\Rightarrow \langle x - y - (Ax - Ay), x - y - (Ax - Ay)\rangle &\leq \|x-y\|^2 + \|Ax - Ay\|^2 \\
\Rightarrow \|x-y\|^2 + \|Ax - Ay\|^2 - 2\langle Ax - Ay, x - y\rangle &\leq \|x-y\|^2 + \|Ax - Ay\|^2 \\
\Rightarrow -2\langle Ax - Ay, x - y\rangle &\leq 0 \\
\Rightarrow \langle Ax - Ay, x - y\rangle &\geq 0 \\
\Rightarrow \langle (I-T)x - (I-T)y, x - y\rangle &\geq 0 \\
\Rightarrow \|x-y\|^2 - \langle Tx - Ty, x - y\rangle &\geq 0 \\
\Rightarrow \langle Tx - Ty, x - y\rangle &\leq \|x-y\|^2
\end{aligned}$$

Nonexpansive $\Rightarrow$ strict pseudo-contraction $\Rightarrow$ pseudo-contraction. However, the following examples show that the converse is not true.

**Example 1.1**: Take $X = \Re^2$, $B = \{x \in \Re^2 : \|x\| \leqslant 1\}$, $B_1 = \{x \in B : \|x\| \leqslant \frac{1}{2}\}$, $B_2 = \{x \in B : \frac{1}{2} \leqslant \|x\| \leqslant 1\}$ . If $x = (a, b) \in X$ we defined $x^\perp$ to be $(b, -a) \in X$. Define $T : B \to B$ by

$$Tx = \begin{cases} x + x^\perp, & x \in B_1 \\ \frac{x}{\|x\|} - x + x^\perp, & x \in B_2 \end{cases}$$

Then, $T$ is Lipschitz and pseudo-contractive but not strictly pseudo-contractive. Example (1.1) is due to Chidume and Mutangadura [12].

**Proof**

Trivially $\langle x, x^\perp\rangle = 0, \|x^\perp\| = \|x\|$, $\langle x^\perp, y\rangle + \langle x, y^\perp\rangle = 0, \langle x^\perp, y^\perp\rangle = \langle x, y\rangle, \|x^\perp - y^\perp\| = \|x - y\|$ $\forall\ x, y \in H$.

We now show that $T$ is imfact Lipschitz. For $x, y \in B_1$ we have

$$\begin{aligned}
\|Tx - Ty\|^2 &= \|x + x^\perp - y - y^\perp\|^2 = \langle x - y + x^\perp - y^\perp, x - y + x^\perp - y^\perp\rangle \\
&= \langle x - y, x - y + x^\perp - y^\perp\rangle + \langle x^\perp - y^\perp, x - y + x^\perp - y^\perp\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle x-y, x-y\rangle + \langle x-y, x^{\perp}-y^{\perp}\rangle + \langle x^{\perp}-y^{\perp}, x-y\rangle + \langle x^{\perp}-y^{\perp}, x^{\perp}-y^{\perp}\rangle \\
&= \|x-y\|^2 + \|x^{\perp}-y^{\perp}\|^2 + \langle x, x^{\perp}\rangle - \langle x, y^{\perp}\rangle - \langle y, x^{\perp}\rangle + \langle y, y^{\perp}\rangle = 2\|x-y\|^2 + 0 \\
&\qquad \Rightarrow \|Tx-Ty\| = \sqrt{2}\|x-y\|
\end{aligned}
$$

Next, for $x, y \in B_2$ we have

$$
\begin{aligned}
\|(\frac{x}{\|x\|} - \frac{y}{\|y\|})\|^2 &= \langle \frac{x}{\|x\|} - \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|}\rangle \\
&= \langle \frac{x}{\|x\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|}\rangle - \langle \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|}\rangle \\
&= \langle \frac{x}{\|x\|}, \frac{x}{\|x\|}\rangle - \langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\rangle - \langle \frac{y}{\|y\|}, \frac{x}{\|x\|}\rangle + \langle \frac{y}{\|y\|}, \frac{y}{\|y\|}\rangle \\
&= \frac{\|x\|^2}{\|x\|^2} - 2\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\rangle + \frac{\|y\|^2}{\|y\|^2} = 2 - 2\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\rangle = 2 - \frac{2\langle x, y\rangle}{\|x\|\|y\|} \\
&= \frac{2}{\|x\|\|y\|}\{\|x\|\|y\| - \langle x, y\rangle\} = \frac{1}{\|x\|\|y\|}\{2\|x\|\|y\| - 2\langle x, y\rangle\} \\
&\quad = \frac{1}{\|x\|\|y\|}\{2\|x\|\|y\| + \|x-y\|^2 - (\|x\|^2 + \|y\|^2)\} \\
&\quad = \frac{1}{\|x\|\|y\|}\{\|x-y\|^2 - (\|x\| - \|y\|)^2\} \\
\Rightarrow \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\|^2 &\leqslant 8\|x-y\|^2 \\
\Rightarrow \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\| &\leqslant \sqrt{8}\|x-y\|
\end{aligned}
$$

Hence, for $x, y \in B_2$ we have

$$
\begin{aligned}
\|Tx-Ty\| &= \|\frac{x}{\|x\|} - x + x^{\perp} - \frac{y}{\|y\|} + y - y^{\perp}\| \\
&\leqslant \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\| + \|x-y\| + \|x^{\perp}-y^{\perp}\| \\
&= \sqrt{8}\|x-y\| + \|x-y\| + \|x-y\| = \sqrt{8}\|x-y\| + 2\|x-y\| \\
\leqslant 3\|x-y\| + 2\|x-y\| &= 5\|x-y\|
\end{aligned}
$$

So that $T$ is Lipschitz on $B_2$. Now let $x$ and $y$ be in the interiors of $B_1$ and $B_2$ respectively. Then there exists $\lambda \in (0,1)$ for which $Z = \lambda x + (1-\lambda)y$. Hence,

$$
\begin{aligned}
\|Tx-Ty\| &\leqslant \|Tx-Tz\| + \|Tz-Ty\| \leqslant \sqrt{2}\|x-z\| + 5\|z-y\| \\
\leqslant 5\|x-y\| + 5\|z-y\| &= 5\|x-y\|
\end{aligned}
$$

Thus $\|Tx - Ty\| \leqslant 5\|x-y\|\ \forall\ x,y \in B$ as required.

We now show that $T$ is a pseudo-contraction. First, we note that we may put $j(x) = x$, since $H$ is Hilbert. For $x,y \in B$ put $\Gamma(x,y) = \|x-y\|^2 - \langle Tx - Ty, x-y\rangle$and, if $x$ and $y$ are both non zero, put $\lambda(x,y) = \frac{\langle x,y\rangle}{\|x\|\|y\|}$. Hence to show that $T$ is a pseudo-contraction, we need to proof that $\Gamma(x,y) \geqslant 0\ \ \forall\ x,y \in B$. We only need examine the following three cases

(i) for $x,y \in B_1$, we have

$$\begin{aligned}
\langle Tx - Ty, x-y\rangle &= \langle x + x^{\perp} - y - y^{\perp}, x-y\rangle \\
&= \langle x-y, x-y\rangle + \langle x^{\perp} - y^{\perp}, x-y\rangle \\
&= \|x-y\|^2 + \langle x^{\perp}, x\rangle - \langle x^{\perp}, y\rangle - \langle y^{\perp}, x\rangle + \langle y^{\perp}, y\rangle \\
&\quad = \|x-y\|^2 + 0 - \{\langle x^{\perp}, y\rangle + \langle y^{\perp}, x\rangle\} + 0 = \|x-y\|^2 \\
\Rightarrow \langle Tx - Ty, x-y\rangle &= \|x-y\|^2
\end{aligned}$$

so that $\Gamma(x,y) = 0$;

(ii)For $x,y \in B_2$ we have

$$\begin{aligned}
\langle Tx - Ty, x-y\rangle &= \langle \frac{x}{\|x\|} - x + x^{\perp} - \frac{y}{\|y\|} + y - y^{\perp}, x-y\rangle \\
&= \langle \frac{x}{\|x\|}, x\rangle - \langle \frac{x}{\|x\|}, y\rangle - \langle x,x\rangle + \langle x,y\rangle + \langle x^{\perp}, x\rangle - \langle x^{\perp}, y\rangle - \langle \frac{y}{\|y\|}, x\rangle \\
&\quad + \langle \frac{y}{\|y\|}, y\rangle + \langle x,y\rangle - \langle y,y\rangle - \langle y^{\perp}, x\rangle + \langle y^{\perp}, y\rangle \\
&= \|x\| - \frac{1}{\|x\|}\langle x,y\rangle - \|x\|^2 + \langle x,y\rangle + 0 - \langle x^{\perp}, y\rangle \\
&\quad - \frac{1}{\|y\|}\langle x,y\rangle + \|y\| + \langle x,y\rangle - \|y\|^2 - \langle y^{\perp}, x\rangle + 0 \\
&= \|x\| + \|y\| - \|x\|^2 - \|y\|^2 - \frac{1}{\|x\|}\langle x,y\rangle + 2\langle x,y\rangle \\
&\quad - \{\langle x^{\perp}, y\rangle + \langle y^{\perp}, x\rangle\} - \frac{1}{\|y\|}\langle x,y\rangle \\
&= \|x\| + \|y\| - \|x\|^2 - \|y\|^2 - \frac{1}{\|x\|}\langle x,y\rangle + 2\langle x,y\rangle - 0 - \frac{1}{\|y\|}\langle x,y\rangle \\
&= \|x\| - \|x\|^2 + \|y\| - \|y\|^2 + \langle x,y\rangle(2 - \frac{1}{\|x\|} - \frac{1}{\|y\|}) \\
&= \|x\| - \|x\|^2 + \|y\| - \|y\|^2 + \frac{\langle x,y\rangle}{\|x\|\|y\|}\{2\|x\|\|y\| - \|y\| - \|x\|\}
\end{aligned}$$

$$= \quad \|x\| - \|x\|^2 + \|y\| - \|y\|^2 + \lambda(x,y)\{2\|x\|\|y\| - \|y\| - \|x\|\}$$

Hence

$$\Gamma(x,y) = 2\|x\|^2 + 2\|y\|^2 - \|x\| - \|y\| - \lambda(x,y)\{4\|x\|\|y\| - \|x\| - \|y\|\}$$

Since

$$\{4\|x\|\|y\| - \|x\| - \|y\|\} \geqslant 0 \quad \forall\ x,y \in B_2$$

We have, for fixed $\|x\|$ and $\|y\|$, $\Gamma(x,y)$ has a minimum when $\lambda(x,y) = 1$.

$$\Rightarrow \Gamma(x,y) \leqslant 2\|x\|^2 + 2\|y\|^2 - 4\|x\|\|y\| = 2(\|x\| - \|y\|)^2$$

Again, we have $\Gamma(x,y) \geqslant 0 \ \ \forall\ x,y \in B_2$ as required.

(iii) For $x \in B_1, y \in B_2$ we have

$$\begin{aligned}
\langle Tx - Ty, x - y\rangle &= \langle x + x^{\perp} - \frac{y}{\|y\|} + y + y^{\perp}, x - y\rangle \\
&= \langle x,y\rangle - \langle x,y\rangle + \langle x^{\perp},x\rangle - \langle x^{\perp},y\rangle - \frac{1}{\|y\|}\langle y,x\rangle \\
&\quad + \frac{1}{\|y\|}\langle y,y\rangle + \langle y,x\rangle - \langle y,y\rangle - \langle y^{\perp},x\rangle + \langle y^{\perp},y\rangle \\
&= \|x\|^2 + 0 - \langle x^{\perp},y\rangle - \frac{1}{\|y\|}\langle x,y\rangle + \|y\| \\
&\quad - \|y\|^2 - \langle y^{\perp},x\rangle + 0 \\
&= \|x\|^2 - \|y\|^2 + \|y\| - [\langle x^{\perp},y\rangle + \langle y^{\perp},x\rangle] \\
&\quad - \frac{1}{\|y\|}\langle x,y\rangle = \|x\|^2 - \|y\|^2 + \|y\| - \frac{1}{\|y\|}\langle x,y\rangle \\
= \|x\|^2 + \|y\| - \|y\|^2 - \frac{1}{\|y\|}\frac{\langle x,y\rangle\|x\|\|y\|}{\|x\|\|y\|} &= \|x\|^2 + \|y\| - \|y\|^2 - \lambda(x,y)\|x\|
\end{aligned}$$

Hence

$$\Gamma(x,y) = 2\|x\|^2 - \|y\| + [\|x\| - 2\|x\|\|y\|]\lambda(x,y)$$

Since $(\|x\| - 2\|x\|\|y\|) \leqslant 0$ for $x \in B_1$ and $y \in B_2$, $\Gamma(x,y)$ has its minimum, for fixed $\|x\|$ and $\|y\|$ when $\lambda(x,y) = 1$. We conclude that

$$\Gamma(x,y) \geqslant 2\|x\|^2 - \|y\| + \|x\| - 2\|x\|\|y\| = (\|y\| - \|x\|)(2\|y\| - 1) \geqslant 0 \ \ \forall x \in B_1, y \in B_2$$

Therefore, $T$ is a pseudo-contraction.

Next, we show that $T$ is not a strictly pseudo-contraction

Let $x, y \in B_1$ be such that $\|x-y\| \neq 0$ { for example $x = (\frac{1}{4}, 0), y = (0,0)\}$ and let $\lambda \in [0,1)$ be arbitrary. Then $\|x-y\| \neq 0$ and

$$\begin{aligned}\|Tx - Ty\|^2 &= \|x + x^\perp - (y - y^\perp)\|^2 = \|x - y + x^\perp - y^\perp\|^2 \\ &= \|x-y\|^2 + \|x^\perp - y^\perp\|^2 + 2\langle x - y, x^\perp - y^\perp\rangle = 2\|x-y\|^2 \qquad (1.5)\end{aligned}$$

Furthermore,

$$\|x - Tx - (y - Ty)\|^2 = \|x^\perp - y^\perp\|^2 = \|x-y\|^2 \qquad (1.6)$$

(1.5) and (1.6) imply:

$$\begin{aligned}\|Tx - Ty\|^2 &= \|x-y\|^2 + \|x - Tx - (y - Ty)\|^2 \\ &> \|x-y\|^2 + \lambda\|x - Tx - (y - Ty)\|^2 \quad \forall\ \lambda \in [0,1).\end{aligned}$$

Therefore, $T$ is not strictly pseudocontractive.

**Example 1.2** Take $X = \Re^1$ and define $T : X \to X$ by $Tx = -3x$. Then, $T$ is a strict pseudocontraction but not nonexpansive mapping.

**Proof**

$$|x - Tx - (y - Ty)|^2 = 16|x-y|^2.$$

Hence,

$$\begin{aligned}|Tx - Ty|^2 &= 9|x-y|^2 = |x-y|^2 + 8|x-y|^2 \\ &= |x-y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2;\end{aligned}$$

so that $T$ is $\lambda-$strictly pseudocontractive with $\lambda = \frac{1}{2}$.

Next, we show that $T$ is not non-expansive.

$$|Tx - Ty| = |-3x + 3y| = |-3(x-y)| = 3|x-y| > |x-y|,\ \forall\ x \neq y$$

Hence $T$ is not non-expansive.

**Remark 1.3** The following example shows that the class of pseudocontractive maps properly contains the class of nonexpansive maps.

It also shows that the class of pseudocontractive maps is more general than the class of strictly pseudocontractive maps.

**Example 1.4**: Let $\Re$ be the reals with the usual norm and $K = [0, 1]$. Let $T : [0, 1] \to [0, 1]$ be defined by

$$Tx = 1 - x^{\frac{2}{3}} \qquad (1.7).$$

We assert that $T$ is not Lipschitz. To see this, let $L > 0$ be arbitrary, $r = \min\{1, \frac{1}{L^3}\}$, and consider $x \in (0, r],\ y = 0$. Then $|x - y| = |x| = x$ and

$$|Tx - Ty| \;=\; |1 - x^{\frac{2}{3}} - 1| = x^{\frac{2}{3}} = x^{\frac{-1}{3}}(x)$$

Since $x < \frac{1}{L^3}$, then $x^{\frac{1}{3}} < \frac{1}{L}$. Thus $\frac{1}{x^{\frac{1}{3}}} > L$ (i.e. $x^{\frac{-1}{3}} > L$). Thus

$$|Tx - Ty| = x^{\frac{-1}{3}}(x) > Lx = L|x| = L|x - y|.$$

Since T is not Lipschitz, it is not nonexpansive. Let $x, y, \in [0, 1]$, and let $x \leq y$. Then $x^{\frac{2}{3}} \leq y^{\frac{2}{3}}$ and $-x^{\frac{2}{3}} \geq -y^{\frac{2}{3}}$. Thus $1 - x^{\frac{2}{3}} \geq 1 - y^{\frac{2}{3}}$, and this yields $Tx \geq Ty$.

It follows that if

$$x - y \leq 0, \text{ then } Tx - Ty \geq 0.$$

Hence

$$\langle Tx - Ty, x - y\rangle \leq 0 \leq t|x - y|^2 \ \ \forall\, t > 0$$

Thus $T$ is pseudocontractive. T is *not strictly pseudocontractive* because it is not *Lipschitz.*

Next, we show that the mapping defined in (1.7) has a fixed point. To see this, let $x$ be the fixed point of the given map. Then we have

$$\begin{aligned} 1 - x^{\frac{2}{3}} &= x \\ \Rightarrow (1 - x)^3 = x^2 & \\ \Rightarrow 1 - 3x + 3x^2 - x^3 &= x^2 \\ \Rightarrow x^3 - 2x^2 + 3x - 1 &= 0 \qquad (1.8) \end{aligned}$$

We want to solve a cubic equation of the form $x^3 + ax^2 + bx + c = 0$

Using Cardano's method of solving a cubic equation, we have $x = m - \frac{a}{3}$ where $a$ is a coefficient of $x^2$

$$x = m + \frac{2}{3} \tag{1.9}$$

Then (1.8) becomes $(m + \frac{2}{3})^3 - 2(m + \frac{2}{3})^2 + 3(m + \frac{2}{3}) - 1 = 0$

$$\Rightarrow m^3 + 3m^2 \times \frac{2}{3} + 3m \times \frac{4}{9} + \frac{8}{27} - 2(m^2 + \frac{4m}{3} + \frac{4}{9}) + 3m + 2 - 1 = 0$$

$$\Rightarrow m^3 + \frac{5m}{3} + \frac{11}{27} = 0 \tag{1.10}$$

Equation (1.10) can be re-written as

$$m^3 + pm + q = 0 \tag{1.11}$$

where

$$p = \frac{5}{3}, q = \frac{11}{27} \tag{1.12}$$

letting $m = u + v$, we re-write the above equation (1.12) as

$$u^3 + v^3 + (u + v)(3uv + p) + q = 0 \tag{1.13}$$

Next, we set $3uv + p = 0$, and equation (1.13) becomes

$$u^3 + v^3 = -q$$

Hence we are left with these equations

$u^3 + v^3 = -q$ and $u^3v^3 = \frac{-p^3}{27}$

Since the above equations are the product and sum of $u^3$ and $v^3$, then there is a quadratic equation with roots $u^3$ and $v^3$. This quadratic equation is $t^2 + qt - \frac{p^3}{27} = 0$ with solutions $u^3 = \frac{-q+\sqrt{q^2+\frac{4p^3}{27}}}{2}$ and $v^3 = \frac{-q-\sqrt{q^2+\frac{4p^3}{27}}}{2}$

But from equation (1.12) $q = \frac{11}{27}, p = \frac{5}{3}$

$$\begin{aligned}
\Rightarrow u^3 &= \frac{\frac{-11}{27} + \sqrt{(\frac{11}{27})^2 + \frac{4\times(\frac{5}{3})^3}{27}}}{2} \\
&= \frac{\frac{-11}{27} + \sqrt{\frac{121}{729} + \frac{500}{729}}}{2} \\
\Rightarrow u^3 &= \frac{-11 + 3\sqrt{69}}{54} \\
\Rightarrow u &= (\frac{-11 + 3\sqrt{69}}{54})^{\frac{1}{3}}
\end{aligned}$$

and

$$\begin{aligned}
v^3 &= \frac{-11 - 3\sqrt{69}}{54} \\
\Rightarrow v &= (\frac{-11 - 3\sqrt{69}}{54})^{\frac{1}{3}}
\end{aligned}$$

Therefore,

$$m = (\frac{-11 + 3\sqrt{69}}{54})^{\frac{1}{3}} + (\frac{-11 - 3\sqrt{69}}{54})^{\frac{1}{3}}$$

But from equation (1.9) $x = m + \frac{2}{3}$

$$\Rightarrow x = (\frac{-11 + 3\sqrt{69}}{54})^{\frac{1}{3}} + (\frac{-11 - 3\sqrt{69}}{54})^{\frac{1}{3}} + \frac{2}{3}$$

a fixed point.

## 1.4 Iterative Algorithms

In this section, we will present several methods for solving fixed point problems. We will focus on iterative methods (we also call them iterative procedures or algorithms) which are given in the form of the following recurrences:

### 1.4.1 The Picard iteration Method

Let $X$ be any set and $T : X \to X$ a self map. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geqslant 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, ...$$

is called the sequence of successive approximations with the initial value $x_0$. It is also known as the Picard iteration.

The theorem below, called the Banach fixed point or the Banach theorem on contractions, is widely applied in various areas of mathematics. The theorem holds for any complete metric space, and hence, in particular, for every closed subset of a Hilbert space.

**Theorem 1.2 (Banach, 1922)** Let $X$ be a complete metric space and $T : X \to X$ be a contraction. Then $T$ has exactly one fixed point $x^* \in X$. Furthermore, for any $x \in X$, the orbit $\{T_n x\}_{n=0}^{\infty}$ converges to $x^*$ at a rate of geometric progression.

The Banach fixed point theorem is a widely applied tool for an iterative approximation of fixed points. Unfortunately, its application is restricted to contractions. We will need, however, appropriate tools for an iterative approximation of fixed points of non-expansive operators $T$ with $F(T) \neq \varnothing$.

Below, we present several classical fixed points theorems.

**Theorem 1.3 (Brouwer, [8])** Let $X \subseteq \Re$ be non empty compact and convex and $T : X \to X$ be continuous. Then $T$ has a fixed point.

The Brouwer fixed point theorem was generalized by Juliusz Schauder.

**Theorem 1.4 (Schauder, [8])** Let $X$ be a non empty compact and convex subset of a Banach space and $T : X \to X$ be continuous. Then $T$ has a fixed point.

For non-expansive operators in Hilbert space $H$ the compactness of $X \subseteq H$ in the Schauder Theorem (1.4) can be replaced by the boundedness and closeness of $X$.

### 1.4.2 Krasnoselskii Iteration Method [6]

For $x_0 \in K$ and $\lambda \in [0, 1]$ the sequence $\{x_n\}_{n=0}^{\infty}$, defined by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2, ...$$

is called Krasnoselkii iterative method and is denoted by $K_n(x_0, \lambda, T)$.

### 1.4.3 The Mann Iteration Process [21]

Mann iteration process is essentially an averaged algorithm which generates a sequence recursively by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, n \geqslant 0 \tag{1.14}$$

where the initial guess $x_0 \in K$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$.

The Mann iteration method has been successfully employed in approximating fixed points (when they exist) of nonexpansive mappings. This success has not carried over to the more general class of pseudo-contractions. If $K$ is a compact convex subset of a Hilbert space and $T : K \to K$ is Lipschitz , then, by Schauder fixed point theorem, $T$ has a fixed point in $K$. All efforts to approximate such a fixed point by means of the Mann sequence when $T$ is also assumed to be pseudo-contractive proved to be abortive.

Hicks and Kubicek [18], gave an example of a discontinuous pseudo-contraction with unique fixed point for which the Mann iteration does not always converge. Borwein and Borwein [7] (proposition 8), gave an example of a Lipschitz map (which is not pseudo-contractive) with a unique fixed point for which the Mann sequence fails to converge. The problem for Lipschitz pseudo-contraction still remained open. This was eventually resolved in the negative by Chidume and Mutangadura [12] in the following example.

**Example 1.4.3.1**

Let $H$ be a real Hilbert space $\Re^2$ under the the usual Euclidean inner product.

If $x = (a, b) \in H$ we define by $x^\perp \in H$ to be $(b, -a)$. Trivially, we have

$$\langle x, x^\perp \rangle = 0, \|x^\perp\| = \|x\|,$$

$$\langle x^\perp, y^\perp \rangle = \langle x, y \rangle, \|x^\perp - y^\perp\| = \|x - y\|$$

and $\langle x^\perp, y \rangle + \langle x, y^\perp \rangle = 0$ for all $x, y \in H$.

We take our closed and bounded convex set $K$ to be the closed unit ball in $H$ and put$K_1 = \{x \in H : \|x\| \leq \frac{1}{2}\}$, $K_2 = \{x \in H : \frac{1}{2} \leq \|x\| \leq 1\}$. We define the map $T : K \to K$ as follows:

$$Tx = \begin{cases} x + x^{\perp}, if x \in K_1; \\ \frac{x}{\|x\|} - x + x^{\perp}, if x \in K_2 \end{cases}$$

Then $T$ is a Lipschitz pseudo-contractive map of a compact convex set into itself with a unique fixed point for which no Mann sequence converges.

We notice that, for $x \in K_1 \cap K_2$, The two possible expressions for $Tx$ coincide and that $T$ is continuous on both of $K_1$ and $K_2$. Hence $T$ is continuous on all of $K$. We now show that $T$ is in fact, Lipschitz. One easily shows that

$\|Tx - Ty\| = \sqrt{2}\|x - y\|$ for $x, y \in K_1$. For $x, y \in K_2$, we have

$$\begin{aligned} \|(\frac{x}{\|x\|} - \frac{y}{\|y\|})\|^2 &= \frac{2}{\|x\|\|y\|}(\|x\|\|y\| - \langle x, y\rangle) \\ &= \frac{1}{\|x\|\|y\|}\{\|x-y\|^2 - (\|x\| - \|y\|)^2\} \\ &\leq \frac{1}{\|x\|\|y\|}2\|x-y\|^2 \leq 8\|x-y\|^2 \end{aligned}$$

Hence, for $x, y \in K_2$, we have

$$\|Tx - Ty\| \leq \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\|$$

$$+\|x - y\| + \|x^{\perp} - y^{\perp}\| \leq 5\|x - y\|,$$

So that $T$ is Lipschitz on $K_2$. Now let $x$ and $y$ be in the interiors of $K_1$ and $K_2$ respectively. Then there exist $\lambda \in (0, 1)$ and $z \in K_1 \cap K_2$ for which $z = \lambda x + (1 - \lambda)y$. Hence

$$\begin{aligned} \|Tx - Ty\| &\leq \|Tx - Tz\| + \|Tz - Ty\| \\ &\leq \sqrt{2}\|x - z\| + 5\|z - y\| \\ &\leq 5\|x - z\| + 5\|z - y\| = 5\|x - y\| \end{aligned}$$

Thus $\|Tx - Ty\| \leq 5\|x - y\| \forall x, y \in K$, as required. The origin is clearly a fixed point of $T$. For $x \in K_1, \|Tx\|^2 = 2\|x\|^2$, and for $x \in K_2, \|Tx\|^2 = 1 + 2\|x\|^2 - 2\|x\|$. From these expressions and from the fact that $Tx = x^{\perp} \neq x$ if $\|x\| = 1$, it is easy to show that the origin is the only fixed point of $T$. We now show that no Mann iteration sequence for $T$

is convergent for any non zero starting point.

First, we show that no Mann sequence converges to the fixed point. Let $x \in K$ be such that $x \neq 0$. Then, in case $x \in K_1$, any Mann iterate of $x$ is actually further away from the fixed point of $T$ than $x$ is. This is because

$\|(1-\lambda)x + \lambda Tx\|^2 = (1+\lambda^2)\|x\|^2 > \|x\|^2$ for $\lambda \in (0,1)$. If $x \in K_2$ then, for any $\lambda \in (0,1)$

$$\begin{aligned}\|(1-\lambda)x + \lambda Tx\|^2 &= \|(\frac{\lambda}{\|x\|} + 1 - 2\lambda)x + \lambda x^{\perp}\|^2 \\ &= [(\frac{\lambda}{\|x\|} + 1 - 2\lambda)^2 + \lambda^2]\|x\|^2 > 0.\end{aligned}$$

more generally, it is easy to see that for the recursion formula (1.2.8), if $x_0 \in K_1$ then $\|x_{n+1}\| > \|x_n\|$ for all integers $n \geqslant 0$, and if $x_0 \in K_2$, then $\|x_{n+1}\| \geqslant \frac{\sqrt{2}}{2}\|x_n\|$ for all integers $n \geqslant 0$. We therefore conclude that, in addition, any Mann iterate of any non zero vector in $K$ is itself non zero. Thus any Mann sequence $\{x_n\}$, starting from a non zero vector, must be infinite. For such a sequence to converge to the origin, $x_n$ would have to lie in the neighbourhood $K_1$ of the origin for all $n > N_0$, for some real $N_0$. This is not possible because, as already established for $K_1$, $\|x_n\| < \|x_{n+1}\|$ for all $n > N_0$.

We now show that no Mann sequence converges to $x \neq 0$. We do this in the form of a general lemma.

**Lemma 1.2** Let $M$ be a non empty closed and convex subset of a real Banach space $E$ and let $S : M \to M$ be any continuous function. If a Mann sequence for $S$ is norm convergent, then the corresponding limit is a fixed point for $S$.

proof. Let $\{x_n\}$ be a Mann sequence in $M$ for $S$, as defined in the recursion formula (1.2.8). Assume, for proof by contradiction, that the sequence converges, in norm, to $x$ in $M$, where $Sx \neq x$. For each $n \in N$ put $\varepsilon_n = x_n - Sx_n - x + Sx$. Since $S$ is continuous, the sequence $\varepsilon_n$ converges to 0. Pick $p \in N$ such that, if $m \geqslant p$ and $n \geqslant p$, then $\|\varepsilon_n\| < \frac{1}{3}\|x - Sx\|$ $\|x_n - x_m\| < \frac{1}{3}\|x - Sx\|$. Pick any positive integer $q$ such that $\sum_{n=p}^{p+q} \alpha_n \geqslant 1$. We have that

$$\|x_p - x_{p+q+1}\| = \|\sum_{n=p}^{p+q}(x_n - x_{n+1})\| = \|\sum_{n=p}^{p+q}\alpha_n(x_n - Sx + \varepsilon_n)\|$$

$$\begin{aligned}
&\geqslant \|\sum_{n=p}^{p+q}\alpha_n(x-Sx)\| - \|\sum_{n=p}^{p+q}\alpha_n\varepsilon_n\| \\
&\geqslant \sum_{n=p}^{p+q}\alpha_n(\|x-Sx\|) - \frac{1}{3}\|x-Sx\|) \geqslant \frac{2}{3}\|x-Sx\|
\end{aligned}$$

The contradiction proves the result.

We now show that $T$ is a peudo-contraction. First, we note that we may put $j(x) = x$, since $H$ is Hilbert. For $x, y \in K$, put $\Gamma(x,y) = \|x-y\|^2 - \langle Tx - Ty, x-y\rangle$ and, if $x$ and $y$ are both non zero, put $\lambda(x,y) = \frac{\langle x,y\rangle}{\|x\|\|y\|}$. Hence to show that $T$ is a pseudo-contraction, we need to proof that $\Gamma(x,y) \geqslant 0$ for all $x, y \in K$. We only need examine the following three cases:

(i) $x, y \in K_1$: An easy computation shows that $\langle Tx - Ty, x - y\rangle = \|x-y\|^2$ so that $\Gamma(x,y) = 0$; thus we are home and dry for this case.

(ii)$x, y \in K_2$: Again, a straight forward calculation shows that

$$\begin{aligned}
\langle Tx - Ty, x-y\rangle &= \|x\| - \|x\|^2 + \|y\| - \|y\|^2 \\
&\quad + \langle x, y\rangle(2 - \frac{1}{\|x\|} - \frac{1}{\|y\|}) \\
&= \|x\| - \|x\|^2 + \|y\| - \|y\|^2 \\
&\quad + \lambda(x,y)(2\|x\|\|y\| - \|x\| - \|y\|).
\end{aligned}$$

Hence $\Gamma(x;y) = 2\|x\|^2 + 2\|y\|^2 - \|x\| - \|y\| - \lambda(x;y)(4\|x\|\|y\| - \|x\| - \|y\|)$

It is not hard to establish that $(4\|x\|\|y\| - \|x\| - \|y\|) \geqslant 0 \forall x, y \in K_2$. Hence, for fixed $\|x\|$ and $\|y\|, \Gamma(x,y)$ has a minimum when $\lambda(x,y) = 1$. This minimum is therefore $2\|x\|^2 + 2\|y\|^2 - 4\|x\|\|y\| = 2(\|x\| - \|y\|)^2$.

Again, we have that $\Gamma(x,y) \geqslant 0 \quad \forall \quad x, y \in K_2$ as required.

(iii) $x \in K_1, y \in K_2$: we have

$\langle Tx - Ty, x - y\rangle = \|x\|^2 + \|y\| - \|y\|^2 - \lambda(x,y)\|x\|$. Hence $\Gamma(x;y) = 2\|y\|^2 - \|y\| + (\|x\| - 2\|x\|\|y\|)\lambda(x;y)$. Since $\|x\| - 2\|x\|\|y\| \leq 0$ for $x \in K_1$ and $y \in K_2, \Gamma(x;y)$ has its minimum, for fixed $\|x\|$ and $\|y\|$ when $\lambda(x;y)=1$. We conclude that

$$\begin{aligned}\Gamma(x,y) &\geqslant 2\|y\|^2-\|y\|+\|x\|-2\|x\|\|y\| \\ &= (\|y\|-\|x\|)(2\|y\|-1)\geqslant 0 \forall x\in K_1, y\in K_2.\end{aligned}$$

This complete the proof.

### 1.4.4 The Ishikawa Iteration Process [19]

In 1974, Ishikawa introduced an iteration scheme which in some sense, is more general than that of Mann and which converges under this setting, to a fixed point of $T$. He proved the following result.

**Theorem 1.4.4.1 (Ishikawa [19])**: If $K$ is a compact convex subset of a Hilbert space $H$, $T:K\to K$ is a Lipschitzian pseudo-contractive map and $x_0$ is any point of $K$, then the sequence $\{x_n\}_{n\geqslant 0}$ converges strongly to a fixed point of $T$, where $x_n$ is defined iteratively for each integer $n\geqslant 0$ by

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n Ty_n, y_n=(1-\beta_n)x_n+\beta_n Tx_n \tag{1.15}$$

where $\{\alpha_n\},\{\beta_n\}$ are sequences of positive numbers satisfying the conditions
(i) $0\leq\alpha_n\leq\beta_n<1$; (ii) $\lim_{n\to\infty}\beta_n=0$; (iii) $\sum_{n\geqslant 0}\alpha_n\beta_n=\infty$.
The iteration method of Theorem 1.4.4.1 which is now referred to as the Ishikawa iterative method has been studied extensively by various authors.The important thing here is that Ishikawa yielded strong convergence (Normal Mann only yields weak convergence). But it is still an open question whether or not this method can be employed to approximate fixed points of Lipschitz pseudo-contractive mapping without the compactness assumption on $K$ or $T$ (see, e.g., [13,22,21]).
In order to obtain a strong convergence theorem for pseudo-contractive mappings without the compactness assumption, Zhou [43] established the hybrid Ishikawa algorithm for Lipschitz pseudo-contractive mappings as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ Z_n = (1-\beta_n)x_n + \beta_n T y_n, \\ K_n = \{z \in K : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n(1-2\alpha_n - L^2\alpha_n^2)\|x_n - Tx_n\|^2\} \\ Q_n = \{z \in K : \langle x_n - z, x_0 - x_n\rangle \geqslant 0\}, \\ x_{n+1} = P_{k_n}\bigcap Q_n x_0, n \geqslant 1. \end{cases}$$

He proved that the sequence $\{x_n\}$ defined by (1.3.0) converges strongly to $P_k x_0$, where $P_k$ is the metric projection from $H$ into $K$.

In [31] Rhoades compared the performance of these two iteration processes and showed that despite they are similar, they may exhibit different behaviours for different classes of nonlinear mappings. In its original form the Ishikawa iteration process does not include Mann process as a special case because of the condition $0 \leq \alpha_n \leq \beta_n \leq 1$. In an effort to have an Ishikawa type iteration process which does include the Mann process as a special case, several authors have modified the inequality condition to read $0 \leq \alpha_n,\beta_n \leq 1$.

In 1995, Liu [23] introduced the Ishikawa and Mann iteration processes with "errors" as follows:

### 1.4.5 Mann Iteration Process with Errors in the Sense of Liu

Let $T : E \to E$ be a given map. Then the Mann iteration process with errors in the sense of Liu is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n + u_n, n \geq 0$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a suitable real sequence in $[0,1]$ and $\{u_n\}_{n=1}^{\infty}$ is a summable sequence in $E$ (*i.e.*, $\sum\limits_{n=1}^{\infty} ||u_n|| < \infty$).

### 1.4.6 Ishikawa Iteration Process with Errors in the Sense of Liu

Let$E$ be a real Banach Space, $T : D(T) \subset E \to E$ and $x_0 \in D(T)$. The Ishikawa iteration process with errors is given by

$$y_n = (1-\beta_n)x_n + \beta_n T x_n + u_n, n \geq 0$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n + v_n, n \geq 0$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are suitable sequences in $[0,1]$ and $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ are summable sequences in $E$.

Several authors have studied these iteration processes with errors and observed that it is unsatisfactory because the conditions

$$\sum_{n=0}^{\infty} ||u_n|| < \infty, \text{ and} \sum_{n=0}^{\infty} ||v_n|| < \infty$$

imply that error terms tend to zero. This is incompatible with the randomness of the occurence of errors. Furthermore, if the operator $T$ is defined on a nonempty convex subset $K$ of a normed linear space $E$, then the Mann and Ishikawa iteration process with errors are not in general well defined.

### 1.4.7 The Agarwal-O'Regan-Sahu Iteration Process

The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in K, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n Tx_n + (1-\beta_n)Ty_n, n \geqslant 1 \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in [0,1] is known as Agar-O'Regan-Sahu [19] iteration scheme.

**Theorem 1.4.7.1 [37]** Let $K$ be a non empty closed convex subset of a real Hilbert space $H$, $S : K \to K$ be non-expansive and $T : K \to K$ be Lipschitz strongly pseudo-contractive mappings respectively such that

$F(S) \cap F(T) = \{x \in K : Sx = x = Tx\} \neq \varnothing$ and satisfying

$\|x - Sy\| \leq \|Sx - Sy\|$,

$\|x - Ty\| \leq \|Tx - Ty\|$ for all $x, y \in K$.

Let $\{\alpha_n\}_{n\geqslant 1}, \{\beta_n\}_{n\geqslant 1} \in [0,1]$ be such that

(i) $\alpha_n \leq \beta_n$

(ii) $\sum_{n\geqslant 1} \alpha_n = \infty$ and

(iii) $\sum_{n\geqslant 1}\beta_n < \infty$.

For arbitrary $x_1 \in K$ let $\{x_n\}_{n\geqslant 1}$ be iteratively defined by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)Sy_n + \alpha_n T x_n, n \geqslant 1. \end{cases}$$

Then $\{x_n\}_n \geqslant 1$ converges strongly to the common fixed point $p$ of $S$ and $T$.

**corollary 1.4.7.2** **[37]** Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T, S : K \to K$ be two Lipschitz pseudo-contractive mappings such that $p \in F(T) \cap F(S) = \{x \in K : Tx = x = Sx\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in [0,1] satisfying $\lim_{n\to\infty}(1-\alpha_n) = 0$. For arbitrary $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)T x_n, n \geqslant 1 \\ x_{n+1} = \alpha_n T x_n + (1-\alpha_n)Sy_n, \end{cases}$$

Then the following are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the common fixed point $q$ of $T$ and $S$.

(b) $\{Tx_n\}_{n=1}^{\infty}$ and $\{Sy_n\}_{n=1}^{\infty}$ are bounded.

## 1.5 ORGANIZATION OF THESIS

We have introduced in Chapter one, various iterations methods for Lipschitz pseudo-contractive maps and some existing results on them. We have also considered various non-linear operators studied in this project.

In Chapter Two (the preliminary section), we presented most of the technical results about convergent sequences of real numbers encountered in Operator Theory. We also considered projection maps and some other results vital to this work.

Chapter Three featured the main results of this project. Using a modified Ishikawa iteration method we proved strong convergence theorem for Lipschitz pseudo-contractive maps in Hilbert spaces. Our result extended the work of Yao and Li [20] from the class of $\lambda$ - strictly pseudo-contractive operators to the more general class of Lipschitz pseudo-contractive operators.

# Chapter 2

# Preliminaries

In this Chapter we present mostly geometric conditions ensuring that convergence is strong. Most of the properties will be established in the frame work of Hilbert spaces. Although a lot of results can be extended to larger classes of spaces, we will only do so in some specific cases, since our aim is to emphasize unity in terms of tools and approach.

## 2.1 Definitions and Technical Results About Convergent Sequences of Real Numbers.

### 2.1.1 Definition (Strong Convergence)

Let $H$ be a Hilbert space. We say that $\{x_n\}$ converges (strongly) to $x$ if $\lim_{n\to\infty} \|x_n - x\| = 0$. This is written $\lim_{n\to\infty} x_n = 0$ or simply $x_n \to x$.

### 2.1.2 Definition (Weak Convergence)

Let $H$ be a Hilbert space. We say that the sequence $\{x_n\}_{n=0}^{\infty}$ of elements of a Hilbert space $H$ converges weakly to $x \in H$ if there is an $x \in H$ such that for every $f \in H^*$, $\lim_{n\to\infty} f(x_n) = f(x)$. We call the point $x$ a weak limit of the sequences $\{x_n\}_{n=0}^{\infty}$ and write $x_n \rightharpoonup x$.

**Proposition 2.1.3** A sequence $\{x_n\}$ in a real Hilbert space $H$ converges weakly to a point $x \in H$ if and only if $\lim_n \langle x_n, z\rangle = \langle x, z\rangle \quad \forall \ z \in H$.

Proof:

Suppose $x_n \rightharpoonup x$, then $f(x_n) \to f(x) \quad \forall \quad f \in H^*$. Let $z \in H$ be arbitrary (but fixed). Then define $g : H \to f = \Re \;\; or \;\; C$ by $g(x) = \langle x, z\rangle$. Then $g \in H^*$, since

$$g(\alpha x + \beta y) = \langle \alpha x + \beta y, z\rangle = \langle \alpha x, z\rangle + \langle \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle = \alpha g(x) + \beta g(y)$$

Also, $\mid g(x) \mid = \mid \langle x, z\rangle \mid \leqslant \|x\|\|z\|$. This implies that $g$ is bounded and $\|g\| \leqslant \|z\|$.

Since $g \in H^*$ and $x_n \rightharpoonup x$ then $g(x_n) \to g(x)$

$\Rightarrow \langle x_n, z\rangle \to \langle x, z\rangle$

$\Leftarrow$) Suppose $\langle x_n, z\rangle \to \langle x, z\rangle \quad \forall \quad z \in H$ then we prove that $x_n \rightharpoonup x$.

Let $f \in H^*$ be arbitrary. Then by the Riez Representation Theorem, there exists a unique $z \in H$ such that $f(x) = \langle x, z\rangle \;\; \forall \;\; x \in H$. Thus $f(x_n) = \langle x_n, z\rangle$ and since $\langle x_n, z\rangle \to \langle x, z\rangle$, we have that $f(x_n) \to f(x)$, which implies that $x_n \rightharpoonup x$

**Definition 2.1.4** A sequence $\{a_n\}$ is said to be monotone increasing if $a_{n+1} \geqslant a_n$ for all $n \geqslant 1$, and monotone decreasing if $a_{n+1} \leq a_n$ for $n \geqslant 1$. We say $\{a_n\}$ is monotone (or monotonic) if it is of one of these two types.

We now state and prove the following important results concerning sequences of real numbers.

**Lemma 2.1.1 [19]** Let $\{a_n\}$ and $\{b_n\}$ be two sequences of a normed space $X$ and $\{t_n\}$ a sequence of real numbers. If the following conditions

(a) $0 \leq t_n \leq t < 1$ and $\sum_{n=1}^{\infty} t_n = \infty$,

(b) $a_{n+1} = (1 - t_n)a_n + t_n b_n$ for all $n \geqslant 1$,

(c) $\limsup_{n\to\infty} \|b_n\| < +\infty$, are satisfied, then $\limsup_{n\to\infty} \|a_n\| \leq \limsup_{n\to\infty} \|b_n\|$.

proof. From (b) we obtain

$a_n = a_1 \prod_{i=1}^{n-1}(1 - t_i) + \sum_{i=1}^{n-1} \prod_{j=i+1}^{n-1}(1 - t_j)t_i b_i$.

Thus we have

$$
\begin{aligned}
\|a_n\| &\leq \|a_1\|\prod_{i=1}^{n-1}(1-t_i)+\sum_{i=1}^{n-1}\prod_{j=i+1}^{n-1}(1-t_j)t_i\|b_i\| \\
&= \|a_1\|\prod_{i=1}^{n-1}(1-t_i)+\sum_{i=1}^{n-1}(\prod_{j=i+1}^{n-1}(1-t_j)-\prod_{j=1}^{n-1}(1-t_j))\|b_i\|.
\end{aligned}
$$

From (c) we have that for each $\varepsilon > 0$ there exists $n_1 \in N$ such that for $n \geqslant n_1$

$\|b_n\| < d + \varepsilon$,

holds, where $d = \limsup_{n\to\infty} \|b_n\|$.

On the other hand, from (a) we have that for each $\varepsilon > 0$ there exists $n_0 \in N$ such that for $n \geqslant n_0 + n_1$

$\prod_{i=n_1}^{n-1}(1-t_i) \leq e^{-\sum_{i=n_1}^{n-1} t_i} < \varepsilon$,

holds, here we use the following inequality $1 + x \leq e^x, x \in \Re$.

Thus for $n \geqslant n_0 + n_1$ we have

$$
\begin{aligned}
\sum_{i=1}^{n-1}(\prod_{j=i+1}^{n-1}(1-t_j)-\prod_{j=i}^{n-1}(1-t_j))\|b_i\| &\leq \|\{b_n\}\|_\infty(\prod_{j=n_1}^{n-1}(1-t_j)-\prod_{j=1}^{n-1}(1-t_j)) \\
&\quad +(d+\varepsilon)(1-\prod_{j=n_1}^{n-1}(1-tj)) \\
&\leq 2\varepsilon\|\{b_n\}\|_\infty+(d+\varepsilon),
\end{aligned}
$$

where $\|\{b_n\}\|_\infty = \sup_{i\in N} \|\{b_n\}\|$.From (c) we have $\|\{b_n\}\|_\infty < \infty$. From all of the above we have $\|a_n\| \leq \varepsilon\|a_1\| + 2\varepsilon\|\{b_n\}\|_\infty(d + \varepsilon)$ for $n \geqslant n_0 + n_1$.

Since $\varepsilon > 0$ is arbitrary we obtain the result.

**Lemma 2.1.2 [44]** Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of non-negative numbers satisfying

$a_{n+1} \leq (1 + \delta_n)a_n + b_n, n \geqslant 1.$

If $\sum_{n=1}^\infty \delta_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$ then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero then $\lim_{n\to\infty} a_n = 0$.

proof. Observe that

$$\begin{aligned}
a_{n+1} &\leq (1+\delta_n)a_n + b_n \\
&\leq (1+\delta_n)[(1+\delta_{n-1})a_{n-1} + b_{n-1}] + b_n \\
&\leq \ldots \leq \prod_{j=1}^{n}(1+\delta_j)a_1 + \prod_{j=1}^{n}(1+\delta_j)\sum_{j=1}^{n} b_j \\
&\leq \ldots \leq \prod_{j=1}^{\infty}(1+\delta_j)a_1 + \prod_{j=1}^{\infty}(1+\delta_j)\sum_{j=1}^{\infty} b_j < \infty.
\end{aligned}$$

Hence $\{a_n\}$ is bounded. Let $M > 0$ be such that $a_n \leq M, n \geqslant 1$. Then

$a_{n+1} \leq (1+\delta_n)a_n + b_n \leq a_n + M\delta_n + b_n = a_n + \gamma_n$

where $\gamma_n = M\delta_n + b_n$. It now follows from lemma 2.1.1 of [37] that $\lim_{n\to\infty} a_n$ exists. Consequently, if $\{a_n\}$ has a subsequence which converges strongly to zero then $\lim_{n\to\infty} a_n = 0$ completing the proof.

**Lemma 2.1.3 [31]** Let $\{a_n\}_{n=1}^{\infty}$ be real sequences of real numbers with $a_n \geq 0 \; \forall \; n$. Let $\liminf a_n = \delta > 0$. Then there exists $n_0$ such that $a_n \geq \frac{\delta}{2} \;\; \forall \;\; n \geq n_0$. Consequently there exists a subsequence $\{a_{n_j}\}_{n=1}^{\infty} \subset \{a_n\}_{n=1}^{\infty}$ such that $a_{n_j} \to \delta$ as $j \to \infty$.

**Proof:** Define $M_n := \inf_{j\geq n} a_j$, then $M_n \leq a_n \forall n$. Now,

$$\delta = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} (\inf_{j\geq n} a_j) = \lim_{n\to\infty} M_n.$$

Hence for $\epsilon = \frac{\delta}{2} > 0$, $\exists n_0$ such that

$$|M_n - \delta| \leq \frac{\delta}{2} \forall n \geq n_0.$$

This implies that

$$\frac{\delta}{2} \leq M_n \leq a_n, \forall n \geq n_0.$$

Consequently, $\{M_n\}_{n=1}^{\infty} \subset \{a_n\}_{n=1}^{\infty}$ and

$$\delta = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} (\inf_{j\geq n} a_j) = \lim_{n\to\infty} M_n.$$

An important application of lemma 2.1.3 can be seen in the following lemma.

**Lemma 2.1.4** [**6**] Suppose $a_n, \alpha_n \geq 0 \ \forall \ n$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n a_n < \infty$. Then $\liminf_{n\to\infty} a_n = 0$.

**Proof:** Suppose that $\liminf_{n\to\infty} a_n = \delta > 0$, then by Lemma 2.1.3 there exists $n_0$ such that $a_n \geq \frac{\delta}{4} \ \forall \ n \geq n_0$ Hence,

$$\alpha_n a_n > \alpha_n \frac{\delta}{4}, \forall \ n \geq n_0.$$

This implies that,

$$\sum_{n=1}^{\infty} \alpha_n a_n > \sum_{n=1}^{\infty} \frac{\delta}{4} \alpha_n = \frac{\delta}{4} \sum_{n=1}^{\infty} \alpha_n = \infty$$

This contradiction implies that $\liminf_{n\to\infty} a_n = 0$.

**Lemma 2.1.5** Let $a_n, b_n \geq 0, \ \forall \ n, \alpha \in [0, 1)$, $\lim\limits_{n\to\infty} b_n = 0$. If $a_{n+1} \leq \alpha a_n + b_n$, then $\lim\limits_{n\to\infty} a_n = 0$

**Proof:**

$$a_{n+1} \leq \alpha a_n + b_n = a_n - (1 - \alpha) a_n + b_n.$$

**Claim :** $\liminf_{n\to\infty} a_n = 0$

**Proof:** Suppose $\liminf_{n\to\infty} a_n = \delta > 0$. By Lemma 2.1.3 $\exists \ n_0$ such that

$$a_n \geq \frac{\delta}{2}, \forall \ n \geq n_0.$$

Also since $\lim_{n\to\infty} b_n = 0$, $\exists \ n_1$ such that

$$b_n \leq \frac{\delta}{4}(1 - \alpha), \forall n.$$

Hence,for $N = \max\{n_0, n_1\}$ we have

$$\begin{aligned} a_{n+1} &\leq a_n - (1 - \alpha)\frac{\delta}{2} + \frac{\delta}{4}(1 - \alpha), \forall n > N \\ &= a_n - \frac{\delta}{4}(1 - \alpha) \\ &\Rightarrow \frac{\delta}{4}(1 - \alpha) \leq a_n - a_{n+1} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \sum_{j=1}^{n} \frac{\delta}{4}(1-\alpha) \leq \sum_{j=1}^{n} a_j - a_{j+1} = a_1 - a_{n+1} \leq a_1 \\ &\Rightarrow n[\frac{\delta}{4}(1-\alpha)] \leq a_1 \; \forall \; n \geq 1 \end{aligned}$$

(contradiction).

Hence $\liminf_{n\to\infty} a_n = 0$. From Lemma 2.1.3, there exists a subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to 0$ as $k \to \infty$. Also, $\lim\limits_{n\to\infty} b_n = 0$ implies that $b_{n_k} \to 0$ as $k \to \infty$ for all subsequences $\{b_{n_k}\}_{k=1}^{\infty} \subset \{b_n\}_{n=1}^{\infty}$. Therefore there exists $k_0$ such that $a_{n_k} < \epsilon$, $b_{n_k} < (1-\alpha)\epsilon$ for all $k \geq k_0$. Hence $a_{n_{k_0}} < \epsilon$, $b_{n_{k_0}} < (1-\alpha)\epsilon$. We now show that,

$$a_{n_{k_0}+m} < \epsilon, \forall m \geq 1.$$

We do this by induction. Observe that $b_{n_{k_0}+m} < (1-\alpha)\epsilon$ for all $m \geq 1$.Now for $m = 1$,

$$\begin{aligned} a_{n_{k_0}+1} &\leq \alpha a_{n_{k_0}} + b_{n_{k_0}} \\ &< \alpha\epsilon + (1-\alpha)\epsilon = \epsilon. \end{aligned}$$

We assume it is true for $m = n$, i.e., $a_{n_{k_0}+n} < \epsilon$ and prove for $m = n+1$. Now,

$$\begin{aligned} a_{n_{k_0}+n+1} &\leq \alpha a_{n_{k_0}+n} + b_{n_{k_0}+n} \\ &< \alpha\epsilon + (1-\alpha)\epsilon = \epsilon \end{aligned}$$

Hence $a_{n_{k_0}+m} < \epsilon$ for all $m \geq 1$ so that $\lim\limits_{n\to\infty} a_n = 0$.

Immediate consequences of Lemma 2.1.5 are the following:

**Corollary 2.1** Let $a_n, b_n \geq 0 \;\; \forall \; n$, $\sum_{n=1}^{\infty} b_n < \infty$. Let $\alpha \in [0,1)$, If $a_{n+1} \leq \alpha a_n + b_n$, then $\lim\limits_{n\to\infty} a_n = 0$.

**Proof:** Obvious from Lemma 2.1.5

**Corollary 2.2** Let $a_n \geq 0 \; \forall \; n$, $\alpha \in [0,1)$. If $a_{n+1} \leq \alpha a_n \;\; \forall \;\; n \geq 1$, then $\lim\limits_{n\to\infty} a_n = 0$.

**Remark 2.1** Corollary 2.1 shows that if we replace $\lim\limits_{n\to\infty} b_n = 0$ with $\sum_{n=1}^{\infty} b_n < \infty$ in Lemma 2.1.5 the result still holds since $\sum_{n=1}^{\infty} b_n < \infty$ implies that $\lim\limits_{n\to\infty} b_n = 0$. However, if $a_n, b_n \geq 0$, $\sum_{n=1}^{\infty} b_n < \infty$, and we replace $\alpha$ with $\{\alpha_n\} \subset [0,1)$, in Lemma 2.1.5 and Corollary 2.1, i.e. $a_{n+1} \leq \alpha_n a_n + b_n$, then $\lim\limits_{n\to\infty} a_n$ may not be zero as can be seen with the following example.

**Example 2.1:** Take $a_n = \frac{n+1}{n}$, $b_n = \frac{1}{n^2+3n+2}$, $\alpha_n = \frac{n^2+3n}{n^2+3n+2}$, $\quad \forall \;\; n \geq 1$. Then $a_{n+1} \leq \alpha_n a_n + b_n$, $\lim_{n\to\infty} b_n = 0$ but $\lim\limits_{n\to\infty} a_n \neq 0$.

**Lemma 2.1.6 ([40]):**

$$Let \;\; \{a_n\}_{n=1}^{\infty} \;\; be \;\; a \;\; sequence \;\; of \;\; non-negative \;\; real \;\; numbers \;\; satisfying$$

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \beta_n + \gamma_n, \quad n \geq 0 \tag{2.1}$$

$$where \;\; \{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \;\; and \;\; \{\gamma_n\}_{n=1}^{\infty} \;\; satisfy \;\; the \;\; conditions:$$

(i) $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$, $\quad \sum \alpha_n = \infty$ or equivalently $\prod\limits_{n=1}^{\infty}(1-\alpha_n) = 0$;

(ii) $\limsup\limits_{n\to\infty} \beta_n \leq 0$;

(iii) $\gamma_n \geq 0, (n \geq 0)$, $\quad \sum\limits_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

**Proof.** For any $\epsilon > 0$, let $N$ be an integer big enough so that

$\beta_n < \frac{\epsilon}{2}, \quad \sum\limits_{n=N}^{\infty} \gamma_n < \frac{\epsilon}{2}, n \geq N.$

Using equation (2.1) above and mathematical induction, we obtain for $n > N$

$$\begin{aligned} a_{n+1} &\leq (1-\alpha_n)a_n + \alpha_n\beta_n + \gamma_n \\ &\leq (1-\alpha_n)[(1-\alpha_{n-1})a_{n-1} + \alpha_{n-1}\beta_{n-1} + \gamma_{n-1}] + \alpha_n\beta_n + \gamma_n \\ &= (1-\alpha_{n-1})(1-\alpha_n)a_{n-1} + \alpha_{n-1}(1-\alpha_n)\beta_{n-1} + (1-\alpha_n)\gamma_{n-1} \end{aligned}$$

$$
\begin{aligned}
& +\alpha_n\beta_n + \gamma_n \\
\leq\ & (1-\alpha_{n-1})(1-\alpha_n)a_{n-1} + \alpha_{n-1}(1-\alpha_n)\beta_{n-1} + \gamma_{n-1} + \gamma_n \\
\leq\ & (1-\alpha_{n-1})(1-\alpha_n)[(1-\alpha_{n-2})a_{n-2} + \alpha_{n-2}\beta_{n-2} + \gamma_{n-2}] \\
& +\alpha_{n-1}(1-\alpha_n)\beta_{n-1} + \gamma_{n-1} + \gamma_n \\
=\ & (1-\alpha_{n-1})(1-\alpha_n)(1-\alpha_{n-2})a_{n-2} + (1-\alpha_{n-1})(1-\alpha_n)\alpha_{n-2}\beta_{n-2} \\
& +(1-\alpha_{n-1})(1-\alpha_n)\gamma_{n-2} + \alpha_{n-1}(1-\alpha_n)\beta_{n-1} + \gamma_{n-1} + \gamma_n \\
\leq\ & (1-\alpha_{n-2})(1-\alpha_{n-1})(1-\alpha_n)a_{n-2} + \alpha_{n-2}(1-\alpha_{n-1})(1-\alpha_n)\beta_{n-2} \\
& +\gamma_{n-2} + \gamma_{n-1} + \gamma_n \\
\vdots & \\
\leq\ & 1-\alpha_N(1-\alpha_{N+1})...(1-\alpha_n)a_N + \alpha_N(1-\alpha_{N+1})(1-\alpha_{N+2})...(1-\alpha_n)\beta_N \\
& +\gamma_N + \gamma_{N+1}... + \gamma_n \\
=\ & \Big(\prod_{k=N}^{n}(1-\alpha_k)\Big)a_N + (1-(1-\alpha_N)(1-\alpha_{N+1})(1-\alpha_{N+2})...(1-\alpha_n))\beta_N \\
& +\sum_{k=N}^{n}\gamma_k \\
\leq\ & \prod_{k=N}^{n}(1-\alpha_k)a_N + \Big(1-\prod_{k=N}^{n}(1-\alpha_k)\Big)\beta_N + \sum_{k=N}^{n}\gamma_k.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
a_{n+1} \ \leq\ & \prod_{k=N}^{n}(1-\alpha_k)a_N + \Big(1-\prod_{k=N}^{n}(1-\alpha_k)\Big)\frac{\epsilon}{2} \\
& +\sum_{k=N}^{n}\gamma_k.
\end{aligned}
$$

Thus $\limsup\limits_{n\to\infty}\ a_{n+1} \leq 0 + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since $\epsilon$ is arbitrary, we have $\limsup\limits_{n\to\infty} a_n = 0$. Thus $\lim\limits_{n\to\infty} a_n = 0$.

**Lemma 2.1.7** Let $\{a_n\}_{n=1}^{\infty}$ be real sequences of real numbers with $a_n \geqslant 0 \ \ \forall \ \ n$. Let $\liminf a_n = \delta > 0$. Then there exists $n_0$ such that $a_n \geqslant \frac{\delta}{n} \ \ \forall \ \ n \geqslant n_0$ . Consequently there exists a subsequence $\{a_{nj}\}_{n=1}^{\infty} \subset \{a_n\}_{n=1}^{\infty}$ such that $a_{nj} \to \delta$ as $j \to \infty$.

proof. Define $M_n = \inf_{j \geqslant n} a_j$, then $M_n \leq a_n \forall n$.

Now, $\delta = \liminf_{n\to\infty} a_n = \lim_{n\to\infty}(\inf_{j\geqslant n} a_j) = \lim_{n\to\infty} M_n$.

Hence for $\varepsilon = \frac{\delta}{2} > 0, \exists n_0$ such that

$|M_n - \delta| \leq \frac{\delta}{2} \quad \forall n \geqslant n_0$.

This implies that

$\frac{\delta}{2} \leq M_n \leq a_n, \forall n \geqslant n_{0.}$

consequently, $\{M_n\}_{n=1}^{\infty} \subset \{a_n\}_{n=1}^{\infty}$ and

$\delta = \liminf_{n\to\infty} a_n = \lim_{n\to\infty}(\inf_{j\geqslant n} a_j) = \lim_{n\to\infty} M_n$.

An important application of lemma 2.1.7 can be seen in the following lemma.

**Lemma 2.1.8 ([31])** Assume $\{a_n\}$ is a sequence of non negative real numbers such that $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, n \geqslant 0$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\Re$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.1.9** Let $\{a_n\}, \{b_n\}, \{t_n\}, \{\delta_n\}$ and $\{\varepsilon_n\}$ be non negative sequences of real numbers such that

(a) $t_n \in [0,1], n \in N \cup \{0\}$, and $t_n \to 0$ as $n \to \infty$;

(b) $\delta_n \to 0$ as $n \to \infty$;

(c) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$;

(d) $\{a_n\}$ is bounded and $\liminf a_n = 0$;

(e) $\lim_{n\to\infty}(a_n - b_n) = 0$;

(f) $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$;

(g) $a_{n+1} \leq a_n(1 - t_n\frac{f_1(b_n)}{f_2(b_n)}) + t_n\delta_n + \varepsilon_n$,

where $f_1$ and $f_2$ are non negative increasing function on $[0,\infty)$ and $f_2(0) > 0$. Then $\lim_{n\to\infty} a_n = 0$.

proof. From the boundedness of the sequences $\{a_n\}$ and (e) it follows that there is a constant $M > 0$ such that $b_n \leq M$ for $n \in N \cup \{0\}$. From the conditions of the theorem we have that for each $\varepsilon > 0$ there is an $n_0 \in N$ such that $a_{n0} < \varepsilon$ and

$$a_n - a_{n+1} > \frac{-\varepsilon}{2}; b_n - a_n > \frac{-\varepsilon}{4}; \sum_{k=n}^{\infty} \varepsilon_k < \varepsilon; \delta_n < \frac{\varepsilon f_1(\frac{\varepsilon}{4})}{2 f_2(M)}$$

for $n \geqslant n_0$.

Using this and (g) we obtain

$$a_{n0+1} \leq \varepsilon + t_{n0}(\frac{\varepsilon f_1(\frac{\varepsilon}{4}) - 2a_{n0} f_1(b_{n0})}{2 f_2(M)}) + \varepsilon_{n0} \tag{2.1}$$

We show that $a_{n0+1} \leq \varepsilon + \varepsilon_{n0}$. Assume the contrary that $a_{n0+1} > \varepsilon + \varepsilon_{n0}$. Then we have $a_{n0} > a_{n0+1} - \frac{\varepsilon}{2} > \frac{\varepsilon}{2}$ and consequently $b_{n0} > a_{n0} - \frac{\varepsilon}{4} > \frac{\varepsilon}{4}$. From this and (2.1) we obtain $a_{n0+1} \leq \varepsilon + \varepsilon_{n0}$, which is a contradiction. Similarly by induction we can obtain
$a_{n0+k} \leq \varepsilon + \sum_{k=o}^{k-1} \varepsilon_{n0+i} < 2\varepsilon$. For all $k \in N$. From this the result follows.

The following lemma is a quantitative version of a lemma due to Moore and Nnoli [2]. In the following, $\Re_+^*$ denotes the set of strictly positive real numbers.

**lemma 2.2.0** Let $\Psi : (0, \infty) \to (0, \infty)$ be an increasing function (formally extended to $0 \ \mathit{by} \ \ \Psi(0) = 0$) and let $\{a_n\}, \{b_n\} \ \ \mathit{and} \ \ \{c_n\}$ be sequences of real non-negative numbers with modulus $N_1 : (0, \infty) \to N$ such that:

$$\forall \varepsilon > 0 \forall n \geqslant N_1(\varepsilon)(c_n \leq b_n \varepsilon) \tag{2.2}$$

Moreover, let $\sum_{n=1}^{\infty} b_n = \infty$ with rate of divergence $N_2 : (0, \infty) \to N$,

$$\forall x \in (0, \infty), (\sum_{i=1}^{N_2(x)} b_i > x) \tag{2.3}$$

If for all $n \geqslant 0$

$$a_{n+1}^2 \leq a_n^2 - b_n\Psi(a_{n+1}) + c_n,$$

then we get a rate of convergence for $\{a_n\}$:

$$\forall \varepsilon > 0 \forall n \geqslant \Phi(N_1, N_2, \varepsilon)(a_n < \varepsilon),$$

where $\Phi : N^{\Re_+^*} \times \Re_+^* \to N$, with

$$\begin{aligned}
\Phi(N_1, N_2, \varepsilon) &= N_2(C) + 1, \\
C &\geqslant \frac{2a_{\bar{N}}^2}{\Psi(\varepsilon)} + \sum_{n=1}^{\bar{N}-1} b_n, \\
\bar{N} = N_1(\frac{\Psi(\varepsilon)}{2}).
\end{aligned}$$

proof. Let $\varepsilon > 0$ be given. We split the proof into two claims:

Claims 1: $\exists n \in [\bar{N}, N_2(C) + 1](a_n < \varepsilon)$

proof. Assume on the contrary that

$\forall n \in [\bar{N}, N_2(C) + 1](a_n \geqslant \varepsilon)$. We first remark that the above interval is non-trivial since for

$\bar{N} \geqslant N_2(C) + 1$, we could get

$\sum_{n=1}^{N_2(C)} b_n \leq \sum_{n=1}^{\bar{N}-1} b_n \leq \sum_{n=1}^{\bar{N}-1} b_n + \frac{2a_{\bar{N}}^2}{\Psi(\varepsilon)} \leq C,$

which is a contradiction to assumption (2.3),i.e., to the rate of divergence of $\sum b_n$. So from this we must have $\bar{N} < N_2(C) + 1$.

By assumption (2.2), we get

$$\forall n \geqslant \bar{N}(C_n b_n^{-1} \leq \frac{1}{2}\Psi(\varepsilon)), \tag{2.4}$$

and so for all $n \in [\bar{N}, N_2(C)]$ it follows that

$$\begin{aligned}
a_{n+1}^2 &\leq a_n^2 - b_n\Psi(a_{n+1}) + C_n \\
&\leq a_n^2 - b_n\Psi(\varepsilon) + \frac{1}{2}\Psi(\varepsilon)b_n \\
&= a_n^2 - \frac{1}{2}\Psi(\varepsilon)b_n.
\end{aligned}$$

Therefore

$\frac{1}{2}\Psi(\varepsilon)b_n \leq a_n^2 - a_{n+1}^2$.

Since this holds for all $n \in [\bar{N}, N_2(C)]$, summing up yields

$$\frac{1}{2}\Psi(\varepsilon)\sum_{n=\bar{N}}^{N_2(C)} b_n \leq a_{\bar{N}}^2 - a_{N_2(C)}^2 \leq a_{\bar{N}}^2 \tag{2.5}$$

Hence after adding $\sum_{n=1}^{\bar{N}-1} b_n$ to (2.5) we get

$\sum_{n=1}^{N_2(C)} b_n \leq \sum_{n=1}^{\bar{N}-1} b_n + \frac{2a_{\bar{N}}^2}{\Psi(\varepsilon)} \leq C$,

which is again a contradiction to (2.3). This proves claim 1. Then for some $n_0 \in [N_1(\frac{\Psi(\varepsilon)}{2}), N_2(C)+1]$ we know that $a_{n_0} < \varepsilon$. We now prove the following claim, which will conclude the result.

Claim 2: $\forall l(a_{n_0+l} < \varepsilon)$.

proof. By induction on $l$. For $l = 0$, this is clear. We prove the induction step by contradiction.

Assume that for some $l \geqslant 0$, we have $a_{n0+l} < \varepsilon$ but $a_{n0+l+1} \geqslant \varepsilon$. Using (2.2) again we get

$$\begin{aligned}
\varepsilon^2 &\leq a_{n0+l+1}^2 \\
&\leq a_{n0+l}^2 - b_{n0+l}\Psi(\varepsilon) + C_{n0+l} \\
&\leq a_{n0+l}^2 - b_{n0+l}\Psi(\varepsilon) + b_{n0+l}\frac{\Psi(\varepsilon)}{2} \\
&< \varepsilon^2 - \frac{b_{n0+l}}{2}\Psi(\varepsilon) \\
&\leq \varepsilon^2,
\end{aligned}$$

which is a contradiction.

## 2.2 Projections onto Convex Set

In this section, we introduce some fundamental notions and results of convex analysis in Hilbert spaces, as well as some metric properties which are vital in our work.

**Definition 2.2.1** Let $K$ be a non-empty subset of $H$. We call

$d_k : H \to \Re$

$x \mapsto \inf_{k\in K} \|x - k\|$

the distance function to $K$.

If $K$ is closed convex then for every $x \in H$ the infimum in the definition of $d_k$ is attained in an unique point, which we denote $P_k x$. This mapping $P_k$, which sends every point in $H$ to its nearest point in $K$, is called the projection onto $K$. $P_k$ is characterized by

$$\|x - P_k x\| \leq \|x - k\| \;\; \forall \;\; k \in K \tag{2.6}$$

**Lemma 2.2.1** Condition (2.6) is equivalent to

$$\langle x - P_k x, P_k x - k\rangle \geqslant 0 \;\; \forall \;\; k \in K \tag{2.7}$$

Proof. (2.7) $\Rightarrow$ (2.6): This is a direct consequence of the identity

$$\begin{aligned} \|x - P_k x\|^2 - \|x - k\|^2 &= \|x - P_k x\|^2 - \|(x - P_k x)' + (P_k x - k)\|^2 \\ &= -2\langle x - P_k x, P_k x - k\rangle - \|P_k x - k\|^2 \end{aligned} \tag{2.8}$$

(2.6) $\Rightarrow$ (2.7): Replacing $k$ in (2.8) by $k' = \lambda k + (1-\lambda)P_k x \in K$ for $\lambda \in (0,1)$ we obtain
$\|x - P_k x\|^2 - \|x - k'\|^2 = -2\lambda\langle x - P_k x, P_k x - k\rangle - \lambda^2\|P_k x - k\|^2$
If (2.6) is satisfied we have
$0 \geqslant -2\lambda\langle x - P_k x, P_k x - k\rangle - \lambda^2\|P_k x - k\|^2$.
Dividing by $\lambda$ and letting $\lambda \to 0$ we get (2.7).

**Lemma 2.2.1 ([1])** Let $K$ be a closed convex subset of a real Hilbert space $H$. Given $x \in H$ and $y \in K$, then $y = P_k x$ if and only if there holds the relation

$$\langle x - y, z - y\rangle \leq 0, \forall z \in K.$$

Proof. ($\Rightarrow$) Let $y = P_k x, z \in K$ and $z_\lambda = y + \lambda(z - y)$ for $\lambda \in (0,1)$. Obviously, $z_\lambda \in K$ because $K$ is convex. By $y = P_k x$ and by the properties of the inner product, we have

$$\begin{aligned}\|x-y\|^2 &\leq \|x-z_\lambda\|^2 = \|x-y-\lambda(z-y)\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle x-y, z-y\rangle + \lambda^2\|z-y\|^2\end{aligned}$$

since $\lambda > 0$, the above inequality yield

$\langle x-y, z-y\rangle \leq \frac{\lambda}{2}\|z-y\|^2$,

and for $\lambda \to 0$ we obtain $\langle x-y, z-y\rangle \leq 0$ in the limit.

($\Leftarrow$) By properties of the inner product and by the fact that $\langle x-y, z-y\rangle \leq 0$ we obtain for any $z \in K$

$$\begin{aligned}\|z-x\|^2 &= \|z-y+y-x\|^2 \\ &= \|z-y\|^2 + \|y-x\|^2 + 2\langle z-y, y-x\rangle \\ &\geqslant \|y-x\|^2,\end{aligned}$$

which by the definition of the metric projection, gives $y = P_k x$.

**Lemma 2.2.2 ([24])** Let $K$ be a closed convex subset of a real Hilbert space $H$. Let $\{x_n\}$ be a sequence in $H$ and $u \in H$. Let $q = P_k u$. If $\{x_n\}$ is such that $\omega_\omega(x_n) \subset K$ and satisfies the conditions $\|x_n - u\| \leq \|u-q\|, \forall n \geqslant 1$. Then $\{x_n\}$ converges strongly to $q$.

**Definition 2.2.2** A normed vector space $X$ is called uniformly convex if for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\|x\| = \|y\| = 1$ and $\|x-y\| \geqslant \varepsilon$ then $\|x+y\| \leq 2(1-\delta)$.

**Example 2.1** Any Hilbert space is uniformly convex.

proof. Let $\varepsilon > 0$ and $x, y \in H$ such that $\|x\| = \|y\| = 1$ and $\|x-y\| \geqslant \varepsilon$. Then the parallelogram law implies

$$\begin{aligned}\|x+y\|^2 &= 2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2 \\ &= 4 - \|x-y\|^2 \\ &\leq 4 - \varepsilon^2\end{aligned}$$

With $\delta = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} > 0$ we obtain thus $\|x+y\| \leq 2(1-\delta)$ and hence $H$ is uniformly convex.

## 2.3 Some Definitions and Results Used in the Main Work

**Definition 2.3.1** Let $K \subset H$ be a closed convex set of a Hilbert space $H$. A mapping $T : K \to H$ is said to be demi-closed (at $y$) if the conditions that $\{x_n\}$ converges weakly to $x$ and that $\{Tx_n\}$ converges strongly to $y$ imply that $x \in K$ and $Tx = y$.

We say that a Banach space $X$ satisfies Opial's condition if $\liminf_{n\to\infty} \|x_n - x\| > \liminf_{n\to\infty} \|x_n - x_0\|$ whenever

$$x_n \rightharpoonup x \neq x_0 \tag{2.9}$$

where $\rightharpoonup$ denotes weak convergence.

**Lemma 2.3.1(29)**. If $x_n \rightharpoonup y \in H, \quad then \quad for \quad any \quad y\prime \in H, y\prime \neq y$, the following inequality holds

$$\liminf_n \|x_n - y\prime\| > \liminf_n \|x_n - y\| \tag{2.10}$$

proof. Let $x_n \rightharpoonup y \ \ and \ \ y\prime \neq y$. Denote $\delta := \|y - y\prime\|^2$ since a weakly convergent sequence is bounded, both lower limits in (2.10) are finite. The properties of the inner product yield

$$\begin{aligned}
\|x_n - y\prime\|^2 &= \|x_n - y + y - y\prime\|^2 \\
&= \|x_n - y\|^2 + \|y - y\prime\|^2 + 2\langle x_n - y, y - y\prime\rangle \\
&= \|x_n - y\|^2 + \delta + 2\langle x_n - y, y - y\prime\rangle
\end{aligned}$$

By assumption, $\langle x_n - y, y - y\prime\rangle \to 0$,

$$\begin{aligned}
consequently, \liminf_n \|x_n - y\prime\|^2 &= \liminf_n \|x_n - y\|^2 + \delta > \liminf_n \|x_n - y\|^2, \\
i.e., \liminf_n \|x_n - y\prime\| > \liminf_n \|x_n - y\|
\end{aligned}$$

Lemma 2.3.1 was proved by Zdzislaw Opial in [31, lemma 1]. The property described in this lemma is known under the name Opial property. A Banach space for which the property holds is called an Opial space. Therefore, lemma 2.3.1 can be formulated as follows: any Hilbert space is an Opial space.

The following lemma, also proved by Opial in [29, Lemma 2], is a consequence of the Opial property.

**Lemma 2.3.2** Let $T : K \to H$ be non-expansive and $y \in K$ be a weak cluster point of a sequence $\{x_n\}_{n=0}^{\infty}$. $If \quad \|Tx_n - x_n\| \to 0, \quad then \;\; y \in F(T)$.

Proof. $x_{nk} \rightharpoonup y$ for a subsequence $\{x_{nk}\}_{n=0}^{\infty} \;\; of \;\; \{x_n\}_{n=0}^{\infty}$ and $\|Tx_n - x_n\| \to 0$. Suppose that $Ty \neq y$. Then, by the triangle inequality and by lemma (2.3.1), we have

$$\begin{aligned}
\liminf_{n\to\infty} \|x_{nk} - y\| &\geqslant \liminf_{n\to\infty} \|Tx_{nk} - Ty\| \\
&= \liminf_{n\to\infty} \|Tx_{nk} - x_{nk} + x_{nk} - Ty\| \\
&\geqslant \liminf_{n\to\infty} (\|x_{nk} - Ty\| - \|Tx_{nk} - x_{nk}\|) \\
&= \liminf_{n\to\infty} \|x_{nk} - Ty\| \\
&> \lim \inf_{n\to\infty} \|x_{nk} - y\|
\end{aligned}$$

A contradiction proves that $y \in F(T)$.

**Definition 2.1.6** We say that an operator $S : K \to H$ is demiclosed at 0 if for any sequence $x_n \rightharpoonup y \in K$ with $Sx_n \to 0$. $Sy = 0$

If we replace the weak convergence $x_n \rightharpoonup y \in K$ by the strong one in definition 2.3.2 then we obtain the definition of the closedness of $S \;\; at \;\; 0$. If $H$ is finite dimensional, then a weakly convergent sequence is convergent. Therefore, the notions of a demiclosed operator and a closed operator coincide in a finite dimensional Hilbert space.

The property expressed in lemma 2.3.2 is known under the name demiclosedness principle (see, e.g., [8, fact 1.2]). This lemma enables us to prove the following useful property of non-expansive mappings.

**proposition 2.3** Let $K \subset H$ be a closed convex set. For every non-expansive mapping $T : K \to H$, the mapping $I - T$ is demi-closed.

Proof. Let $\{x_n\} \subset K$ be a sequence which satisfies $x_n \rightharpoonup x_0$ and $(x_n - Tx_n) \to y_0$. Suppose that $y_0 \neq x_0 - Tx_0$. Then we have by Opial's condition

$$
\begin{aligned}
\liminf_{n\to\infty} \|x_n - y_0 - Tx_0\| &> \liminf_{n\to\infty} \|x_n - x_0\| \\
&\geqslant \liminf_{n\to\infty} \|Tx_n - Tx_0\| \\
&\geqslant \liminf_{n\to\infty} \|x_n - y_0 - Tx_0\|
\end{aligned}
$$

which yields a contradiction. Hence $y_0 = (I - T)x_0$.

As a direct consequence if $y_0 = 0$ we obtain the so-called demiclosedness principle:

## 2.4 Corollary (Demiclosedness Principle)

Let $K \subset H$ be a closed convex set and $T : K \to H$ a non-expansive mapping. If $\{x_n\}$ is a sequence in $K$ and $x \in K$ then

$\{x_n \rightharpoonup x, x_n - Tx_n \to 0\} \Rightarrow x \in F(T)$.

It is well known that if $T : K \to K$ is a strict contraction mapping, i.e. if there exists $k \in (0, 1)$ such that

$\|Tx - Ty\| \leq k\|x - y\| \;\; \forall \;\; x, y \in K$, then $T$ has a unique fixed point.

The following example shows that, in general, this is not true for non-expansive mappings.

**Example 2.4.1** Let $H = l^2$. Let $K = l^2$ and define $T : K \to K$ by $Tx = (1, x_2, x_2, ...)$ for $x = (x_1, x_2, ...)$. It is clear that $T$ is non-expansive. But $T$ has no fixed points since $(1, 1, 1, ...) \notin l^2$

Nevertheless, if $K$ is closed and bounded there are fixed points.

The following result can be found in [35, proposition 5.3]

**Proposition 2.4.2** The subset of fixed points of a non-expansive operator $T : X \to H$ is closed and convex.

Proof. (cf. [10, proposition 5.3]) Let $x_n = F(T)$ and $x_n \to x$. We have $x \in X$ *because* $X$ is closed. By continuity of $T$,

$$x = \lim x_n = \lim Tx_n = Tx.$$

i.e., $F(T)$ is a closed subset. Now we show the convexity of $F(T)$. Let $x, y \in F(T), x \neq y \;\; and \;\; z = (1-\lambda)x + \lambda y \;\; for \;\; \lambda \in (0,1)$. By non-expansivity of $T$ and by the positive homogeneity of the norm we have

$$\|x - Tz\| = \|Tx - Tz\| \leq \|x - z\| = \lambda\|x - y\| \tag{2.11}$$

and

$$\|Tz - y\| = \|Tz - Ty\| \leq \|z - y\| = (1-\lambda)\|x - y\| \qquad (2.12).$$

Now, the triangle inequality yields

$$\begin{aligned} \|x - y\| &\leq \|x - Tz\| + \|Tz - y\| \\ &\leq \lambda\|x - y\| + (1-\lambda)\|x - y\| \\ &= \|x - y\|. \end{aligned}$$

Consequently, $\|x - y\| = \|x - Tz\| + \|Tz - y\|$.

By strict convexity of the norm, the vectors $x - Tz \;\; and \;\; Tz - y$ are positive linearly dependent. Therefore, $\alpha(x - Tz) + \beta(y - Tz) = 0 \;\; for \;\; some \;\; \alpha, \beta \geqslant 0.$.

Since $x \neq y$, it follows that $\alpha + \beta > 0$, and hence, $Tz = \frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y$. Now, the non-expansivity of $T$ and inequality (2.11) and (2.12) yield

$$\begin{aligned} \frac{\beta}{\alpha+\beta}\|x - y\| &= \|x - Tz\| = \|Tx - Tz\| \\ &\leq \|x - z\| = \lambda\|x - y\| \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \frac{\alpha}{\alpha+\beta}\|x - y\| &= \|Tz - y\| = \|Tz - Ty\| \\ &\leq \|z - y\| = (1-\lambda)\|x - y\| \end{aligned} \tag{2.14}$$

If at least one inequality in (2.13) and (21.4) is strict, then by summing up (2.13) and (2.14) we would obtain a contradiction. therefore, $\frac{\beta}{\alpha+\beta} = \lambda \;\; and \;\; \frac{\alpha}{\alpha+\beta} = 1-\lambda, \;\; consequently \;\; Tz = (1-\lambda)x + \lambda y = z.$

**Lemma 2.3.3 ([25])** Let $H$ be a real Hilbert space. $K$ a closed convex subset of $H$ and $T : K \to K$ a continuous pseudo-contractive mapping, then

(i) $F(T)$ is closed convex subset of $K$.

(ii) $I - T$ is demi-closed at zero, i.e., if $\{x_n\}$ is a sequence in $K$ such that $x_n \rightharpoonup z$ and $(I-T)x_n \to 0$, then $(I-T)z = 0$.

**Lemma 2.3.4** Let $H$ be a real Hilbert space. Then there holds the following well known results:

(i) $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \forall x, y \in H$;

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle \forall x, y \in H$.

**Lemma 2.3.5 ([34])** Let $H$ be a real Hilbert space. Then for all $x, y \in H, \alpha \in [0,1]$ the following equality holds:

$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$.

Proof.

$$\begin{aligned}
\|\alpha x + (1-\alpha)y\|^2 &= \alpha^2\|x\|^2 + (1-\alpha)^2\|y\|^2 + 2\alpha(1-\alpha)Re\langle x, y\rangle \\
&= \alpha^2\|x\|^2 + (1-\alpha)^2\|y\|^2 + \alpha(1-\alpha)[\|x\|^2 + \|y\|^2 - \|x-y\|^2] \\
&= \alpha^2\|x\|^2 + (1-\alpha)^2\|y\|^2 + \alpha(1-\alpha)\|x\|^2 + \alpha(1-\alpha)\|y\|^2 \\
&\quad -\alpha(1-\alpha)\|x-y\|^2
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 &= \alpha^2\|x\|^2 + \alpha(1-\alpha)\|x\|^2 \\
&\quad +(1-\alpha)^2\|y\|^2 + \alpha(1-\alpha)\|y\|^2 \\
&= \alpha\|x\|^2 + (1-\alpha)\|y\|^2.
\end{aligned}$$

**Lemma 2.3.6 ([24])** Let $H$ be a real Hilbert space. If $\{x_n\}$ is a sequence in $H$ weakly convergent to $z$, then $\limsup_{n\to\infty}\|x_n - y\|^2 = \limsup_{n\to\infty}\|x_n - z\|^2 + \|z-y\|^2 \forall y \in H$.

**Lemma 2.3.8** Let $K$ be a nonempty convex subset of a real Hilbert space $H$. Let $T_i : K \to K, i = 1, 2, ..., N$, be a finite family of Lipschitz pseudocontractive mappings with constants $L_i$, respectively. Let $T = \theta_1 T_1 + \theta_2 T_2 + ...\theta_N T_N$ where $\theta_1 + \theta_2 + ... + \theta_N = 1$.

Then $T$ is Lipschitz pseudocontractive mapping on $K$.

Proof. Let $x, y \in K$. Then

$$\begin{aligned}\langle Tx - Ty, x - y\rangle &= \theta_1\langle T_1x - T_1y, x - y\rangle \\ &\quad +\theta_2\langle T_2x - T_2y, x - y\rangle + ... + \theta_N\langle T_Nx - T_Ny, x - y\rangle \\ &\leq \theta_1\|x - y\|^2 + \theta_2\|x - y\|^2 + ... + \theta_N\|x - y\|^2 \\ &= \|x - y\|^2.\end{aligned}$$

Hence $T$ is pseudocontractive. Moreover, since

$$\begin{aligned}\|Tx - Ty\| &= \|(\theta_1T_1 + \theta_2T_2 + ... + \theta_NT_N)x - (\theta_1T_1 + \theta_2T_2 + ... + \theta_NT_N)y\| \\ &\leq \theta_1\|T_1x - T_1y\|^2 + \theta_2\|T_2x - T_1y\|^2 + ... + \theta_N\|x - y\|^2 \\ &\leq L\|x - y\|^2\end{aligned}$$

where $L := max\{L_i : i = 1, 2, ..., N\}$, we get that $T$ is L-Lipschitz. The proof is complete.

**Lemma 2.3.9**: Let $H$ be a real Hilbert space, and let $K$ be a nonempty closed convex subset of $H$. Let $P_k$ be the convex projection onto $K$. Then, convex projection is characterized by the following relations;

(i) $x^* \in K$ satisfies $x^* = P_k(x) \Leftrightarrow x^* \in K$ satisfies $\langle x - x^*, y - x^*\rangle \leq 0$, for all $y \in K$

(ii) $P_k$ is firmly nonexpansive with F$(P_k) = k$

(iii) $P_k$ is nonexpansive

(iv) Projection $P$, restricted to $K$ is identity (i.e., $P|_k = I$)

(v) $P_k$ is idempotent on $K$.

**Proof of Lemma 2.3.9**

(i) $\|x - P_kx\| \leq \|x - y\|$, for all $y \in K$

Let $y = (1 - \lambda)P_kx + \lambda u, \lambda \in (0, 1), u \in K$. Then, $(1 - \lambda)P_kx + \lambda u \in K$, for all $u \in K$. Furthermore,

$$\begin{aligned}
\|x - P_k x\|^2 &\leq \|x - [(1-\lambda)P_k x + \lambda u]\|^2 \\
&= \|x - P_k x + \lambda(P_k x - u)\|^2 \\
&= \|x - P_k x\|^2 + 2\lambda\langle x - P_k x, P_k x - u\rangle + \lambda^2\|P_k x - u\|^2.
\end{aligned}$$

Thus, $\langle x - P_k x, u - P_k x\rangle \leq \lambda\|P_k x - u\|^2$.

Letting $\lambda \to 0$, we obtain

$$\langle x - P_k x, u - P_k x\rangle \leq 0, \quad \text{for all u} \in \text{ K}$$

(ii) Let $x \in H$, then $z = P_k x$ if and only if $\langle x - z, z - y\rangle \geq 0$ , for all $y \in K$. In particular,

$$\langle x - P_k x, P_k x - P_k y\rangle \geq 0 \tag{2.16}$$

In a similar argument, $z = P_k y$ if and only if $\langle y - P_k y, P_k y - P_k x\rangle \geq 0$ , for all $x \in K$. That is,

$$\langle P_k y - y, P_k x - P_k y\rangle \geq 0 \tag{2.16}$$

Adding equation (2.15) to equation (2.16) yields

$$\langle x - P_k x + P_k y - y, P_k x - P_k y\rangle \geq 0.$$

This implies that

$$\langle x - y, P_k x - P_k y\rangle - \langle P_k x - P_k y, P_k x - P_k y\rangle \geq 0$$

and so

$$\langle x - y, P_k x - P_k y\rangle \geq \|P_k x - P_k y\|^2. \tag{2.17}$$

This shows that $P_k$ is firmly nonexpansive.

(iii) By Cauchy-Schwartz inequality and from equation (2.17), we have

$$\begin{aligned}
\|P_k x - P_k y\|^2 &\leq \langle x - y, P_k x - P_k y\rangle \\
&\leq |\langle x - y, P_k x - P_k y\rangle| \\
&\leq \|x - y\|\|P_k x - P_k y\|.
\end{aligned}$$

It follows that,

$$\|P_k x - P_k y\| \leq \|x - y\|, \quad \forall\ x, y \in H$$

(iv) Let $x \in H$, we have that

$$\|x - P_k x\| = \inf_{y \in K} \|x - y\| \leq \|x - y\|, \quad \forall\ y \in K. \tag{2.18}$$

If $x \in K$, and in particular if $y = x$, then equation (2.18) becomes

$$\|x - P_k x\| = 0.$$

Thus, $P_k x = x$ and this implies that $P_k$ is an identity on $K$.

(v) Let $x \in H$, and let $P_k x = z$. Then,

$$P_k^2 x = P_k(P_k x) = P_k z = z,$$

since $z \in K$. Therefore $P_k^2 x = P_k x$ and so $P_k^2 = P_k$.

**Lemma 2.4.0** If $x_n \rightharpoonup x \in H$, then $\liminf_n \|x_n\| \geqslant \|x\|$.

Proof. Let $x_n \rightharpoonup x$. The lemma is clear for $x = 0$. Let now assume $x \neq 0$. By the Cauchy Schwartz inequality, we have

$\liminf_n \|x\|.\|x_n\| \geqslant \liminf \langle x, x_n \rangle = \|x\|^2$,

i.e., $\liminf_n \|x_n\| \geqslant \|x\|$.

Note that a sequence which is weakly convergent needs not to converge strongly.

# Chapter 3

# Weak and Strong Convergence of an Iterative Algorithm for Lipschitz Pseudo-Contractive Maps in Hilbert Spaces

## 3.1 Main Result

In Chapter 2, we presented mostly geometric, conditions ensuring that the convergence is strong.

In this Chapter, we introduce a modified Ishikawa iteration for Lipschitz pseudocontractive map in real Hilbert spaces.

**Algorithm 3.1.1** Let $H$ be a real Hilbert space and $K$ be a closed convex subset of $H$. Let $T : K \to K$ be a Lipschitz pseudo-contractive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0, 1)$. For given $x_1 \in K$, let $\{x_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = P_k[(1 - \alpha_n - \gamma_n)x_n + \gamma_n T y_n]; \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, n \geq 1 \end{cases} \tag{3.0}$$

**Theorem 3.1.2** Let $H$ be a real Hilbert space and $K$ be a closed convex subset of $H$. Let $T : K \to K$ be a L-Lipschitz pseudo-contractve mapping such that $F(T) \neq \emptyset$. Assume the sequences $\{\alpha_n\}, \{\gamma_n\}, \{\beta_n\} \in (0, 1)$ satisfying

(i) $\beta_n(1 - \alpha_n) > \gamma_n \forall n \geq 1$ (ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma\alpha_n = \infty$

(iii) $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{[\sqrt{1+L^2}+1]}$ for all $n \geq 1$. Then the sequence $\{x_n\}$ generated by (3.0) strongly converges to a fixed point of $T$.

**Proof of Theorem 3.1.2**

Since $F(T) \neq \emptyset$, we can take $p \in F(T)$. From (3.0) we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|P_k[(1-\alpha_n-\gamma_n)x_n + \gamma_n T y_n] - p\| \\
&\leq \|(1-\alpha_n-\gamma_n)x_n + \gamma_n T y_n - p\| \\
&= \|(1-\alpha_n-\gamma_n)(x_n - p) + \gamma_n(T y_n - p) - \alpha_n p\| \\
&\leq \|(1-\alpha_n-\gamma_n)(x_n - p) + \gamma_n(T y_n - p)\| \\
&\quad +\alpha_n\|p\| \qquad (3.1)
\end{aligned}$$

Now consider

$$\begin{aligned}
\|(1-\alpha_n-\gamma_n)(x_n - p) + \gamma_n(T y_n - p)\|^2 &= \|(1-\alpha_n)[(1-\gamma_n)(x_n - p) + \gamma_n(T y_n - p)] \\
&\quad +\alpha_n[-\gamma_n x_n + \gamma_n T y_n]\|^2
\end{aligned}$$

By Lemma 2.3.5 we have

$$\begin{aligned}
&= (1-\alpha_n)\|(1-\gamma_n)(x_n - p) + \gamma_n(T y_n - p)\|^2 \\
&\quad +\alpha_n\|\gamma_n(T y_n - x_n)\|^2 - \alpha_n(1-\alpha_n)\|x_n - p\|^2 \\
&= (1-\alpha_n)[(1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|T y_n - p\|^2 \\
&\quad -\gamma_n(1-\gamma_n)\|x_n - T y_n\|^2] + \alpha_n\gamma_n^2\|T y_n - x_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - p\|^2 \\
&= (1-\alpha_n)\|x_n - p\|^2 - \gamma_n(1-\alpha_n)\|x_n - p\|^2 \\
&\quad +\gamma_n(1-\alpha_n)\|T y_n - p\|^2 \\
&\quad -\gamma_n(1-\gamma_n)(1-\alpha_n)\|x_n - T y_n\|^2 \\
&\quad +\alpha_n\gamma_n^2\|T y_n - x_n\|^2 - \alpha_n(1-\alpha_n)\|x_n - p\|^2 \\
&\leq (1-\alpha_n)^2\|x_n - p\|^2 - \gamma_n(1-\alpha_n)\|x_n - p\|^2 \\
&\quad +\gamma_n(1-\alpha_n)[\|y_n - p\|^2 + \|y_n - T y_n\|^2] \\
&\quad -\gamma_n(1-\alpha_n-\gamma_n)\|T y_n - x_n\|^2 \qquad (3.2)
\end{aligned}$$

But

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(Tx_n - p)\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|Tx_n - p\|^2 \\
&\quad -\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad +\beta_n\|x_n - Tx_n\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\
&= \|x_n - p\|^2 + \beta_n^2\|x_n - Tx_n\|^2 \qquad (3.3)
\end{aligned}$$

Also,

$$\begin{aligned}
\|y_n - Ty_n\|^2 &= \|(1-\beta_n)(x_n - Ty_n) + \beta_n(x_n - Ty_n\|^2 \\
&= (1-\beta_n)\|x_n - Ty_n\|^2 + \beta_n\|Tx_n - Ty_n\|^2 \\
&\quad -\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\
&\leq (1-\beta_n)\|x_n - Ty_n\|^2 + \beta_n L^2\|x_n - y_n\|^2 \\
&\quad -\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\
&\leq (1-\beta_n)\|x_n - Ty_n\|^2 + \beta_n^3 L^2\|x_n - Tx_n\|^2 \\
&\quad -\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\
&= (1-\beta_n)\|x_n - Ty_n\|^2 \\
&\quad +\beta_n[L^2\beta_n^2 + \beta_n - 1]\|x_n - Tx_n\|^2 \qquad (3.4)
\end{aligned}$$

substituting (3.3),(3.4) in (3.2) we have,

$$\begin{aligned}
\|(1-\alpha_n-\gamma_n)(x_n - p) + \gamma_n(Ty_n - p)\|^2 &= (1-\alpha_n)^2\|x_n - p\|^2 - \gamma_n(1-\alpha_n)\|x_n - p\|^2 \\
&\quad +\gamma_n(1-\alpha_n)[\|x_n - p\|^2 + \beta_n^2\|x_n - Tx_n\|^2] \\
&\quad +(1-\beta_n)\|x_n - Ty_n\|^2 \\
&\quad +\beta_n(L^2\beta_n^2 + \beta_n - 1)\|x_n - Tx_n\|^2]
\end{aligned}$$

$$
\begin{aligned}
& -\gamma_n(1-\gamma_n)(1-\alpha_n)\|x_n - Ty_n\|^2 + \alpha_n\gamma_n^2\|Ty_n - x_n\|^2 \\
= \; & (1-\alpha_n)^2\|x_n - p\|^2 \\
& -\gamma_n[\beta_n(1-\alpha_n) - \gamma_n]\|x_n - Ty_n\|^2 \\
& +\gamma_n\beta_n(1-\alpha_n)[L^2\beta_n^2 + 2\beta_n - 1]\|x_n - Tx_n\|^2 \\
= \; & (1-\alpha_n)^2\|x_n - p\|^2 \\
& -\gamma_n[\beta_n(1-\alpha_n) - \gamma_n]\|x_n - Ty_n\|^2 \\
& -\gamma_n\beta_n(1-\alpha_n)[1 - L^2\beta_n^2 - 2\beta_n]\|x_n - Tx_n\|^2
\end{aligned}
$$

Therefore, by condition (iii) and (i) we have that $1 - 2\beta_n - L^2\beta_n^2 > 0$ and $\beta_n(1-\alpha_n) > \gamma_n$

$$
\begin{aligned}
\Rightarrow \|(1-\alpha_n-\gamma_n)(x_n - p) + \gamma_n Ty_n\|^2 \;&\leq\; (1-\alpha_n)^2\|x_n - p\|^2 \\
&\leq\; (1-\alpha_n)\|x_n - p\| \qquad (3.5)
\end{aligned}
$$

Combining (3.5) and (3.1) we have

$$
\begin{aligned}
\|x_{n+1} - p\| \;&\leq\; (1-\alpha_n)\|x_n - p\| + \alpha_n\|p\| \qquad (3.6) \\
&\leq\; max\{\|x_n - p\|, \|p\|\}
\end{aligned}
$$

By induction, we have

$$
\|x_{n+1} - p\| \;\leq\; max\{\|x_0 - p\|, \|p\|\},
$$

which implies that $\{x_n\}$ is bounded.

Furthermore, from (3.0), Lemma 2.3.3, and following the methods above, we get that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \;&=\; \|P_k[(1-\alpha_n-\gamma_n)x_n + \gamma_n Ty_n] - p\| \\
&\leq\; \|x_n - p - \gamma_n(x_n - Ty_n) - \alpha_n x_n\|^2 \\
&\leq\; \|x_n - p - \gamma_n(x_n - Ty_n)\|^2 - 2\alpha_n\langle x_n, x_{n+1} - p\rangle \\
&=\; \|(1-\gamma_n)x_n + \gamma_n Ty_n - p\|^2 - 2\alpha_n\langle x_n, x_{n+1} - p\rangle \qquad (3.7)
\end{aligned}
$$

Now, consider

$$
\|\gamma_n Ty_n + (1-\gamma_n)x_n - p\|^2 \;=\; \gamma_n\|Ty_n - p\|^2 + (1-\gamma_n)\|x_n - p\|^2 - \gamma_n(1-\gamma_n)\|Ty_n - x_n\|^2
$$

From (1.1.3),(3.3) and (3.4) we have

$$\begin{aligned}
\|\gamma_n Ty_n + (1-\gamma_n)x_n - p\|^2 &\leq \gamma_n[\|y_n - p\|^2 + \|y_n - Ty_n\|^2] + (1-\gamma_n)\|x_n - p\|^2 \\
&\quad -\gamma_n(1-\gamma_n)\|Ty_n - x_n\|^2 \\
&\leq \gamma_n[\|x_n - p\|^2 + \beta_n^2\|x_n - Tx_n\|^2] + \gamma_n[(1-\beta_n)\|x_n - Ty_n\|^2 \\
&\quad +\beta_n(L^2\beta_n^2 + \beta_n - 1)\|x_n - Tx_n\|^2] + (1-\gamma_n)\|x_n - p\|^2 \\
&\quad -\gamma_n(1-\gamma_n)\|Ty_n - x_n\|^2 \\
&= \|x_n - p\|^2 - \gamma_n\beta_n[1 - 2\beta_n - L^2\beta_n^2]\|x_n - Tx_n\|^2 \\
&\quad +\gamma_n(\gamma_n - \beta_n)\|x_n - Ty_n\|^2
\end{aligned}$$

Since from (iii), we have that $(\gamma_n - \beta_n) \leq 0$ and $1 - 2\beta_n - L^2\beta_n^2 > 0 \forall n \geq 1$, we have that

$$\|\gamma_n Ty_n + (1-\gamma_n)x_n - p\|^2 \leq \|x_n - p\|^2 - \gamma_n\beta_n[1 - 2\beta_n - L^2\beta_n^2]\|x_n - Tx_n\|^2 \tag{3.8}$$

substituting (3.8) in (3.7) we have that,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \gamma_n\beta_n[1 - 2\beta_n - L^2\beta_n^2]\|x_n - Tx_n\|^2 - 2\alpha_n\langle x_n, x_{n+1} - p\rangle \tag{3.9}$$

Since $\{x_n\}$ is bounded, so there exists a constant

$$M \geq 0 \ \ such \ \ that \ \ -2\langle x_n, x_{n+1}\rangle \leq M \ \forall \ n \geq 0$$

Consequently, from (3.9), we get

$$\|x_{n+1} - p\|^2 - \|x_n - p\|^2 + \gamma_n\beta_n[1 - 2\beta_n - L^2\beta_n^2]\|x_n - Tx_n\|^2 \leq M\alpha_n \tag{3.10}$$

We now consider the following two cases:

**Case 1:** Suppose that there exists $n_0 \in N$ such that $\{\|x_n - p\|\}$ is non increasing. Then, we get that $\{\|x_n - p\|\}$ is convergent. Clearly we have that $\|x_{n+1} - p\|^2 - \|x_n - p\|^2 \to 0$. This together with (ii) and (3.10) imply that

$$\|x_n - Tx_n\| \to 0 \tag{3.11}$$

By lemma 2.3.2 and (3.11), it is easy to see that $\omega_\omega(x_n) \subset F(T)$, where

$$\omega_\omega(x_n) = \{x : \exists x_{ni} \rightharpoonup x\}$$

is the weak $\omega-$limit set of $\{x_n\}$. This implies that $\{x_n\}$ converges weakly to a fixed point $x^*$ of $T$. Indeed, if we take $x^*, \bar{x} \in \omega_\omega(x_n)$ and let $\{x_{ni}\}$ and $\{x_{mj}\}$ be sequences in $\{x_n\}$ such that $x_{ni} \rightharpoonup x^*$ and $x_{mj} \rightharpoonup \bar{x}$, respectively.

Since $\lim_{n\to\infty} \|x_n - z\|$ exists for $z \in F(T)$. Therefore by lemma 2.3.6, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x^*\|^2 &= \lim_{j\to\infty} \|x_{mj} - x^*\|^2 \\
&= \lim_{j\to\infty} \|x_{mj} - \bar{x}\|^2 + \|\bar{x} - x^*\|^2 \\
&= \lim_{i\to\infty} \|x_{ni} - x^*\|^2 + 2\|\bar{x} - x^*\|^2 \\
&= \lim_{n\to\infty} \|x_n - x^*\|^2 + 2\|\bar{x} - x^*\|^2
\end{aligned}$$

Hence, $\bar{x} = x*$.

Next, we prove that $\{x_n\}$ strongly converges to $x^*$. Let $z_n = \gamma_n T y_n + (1-\gamma_n)x_n$. Then from (3.0) we get that $x_{n+1} = P_k[z_n - \alpha_n x_n], \forall n \geq 0$

It follows that

$$x_{n+1} = P_k[(1-\alpha_n)z_n + \alpha_n(z_n - x_n)] \tag{3.12}$$

At the same time, we note

$\|z_n - x^*\|^2 = \|x_n - x^* - \gamma_n(x_n - Ty_n)\|^2$

using the same approach as in (3.8) we have

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \gamma_n\beta_n[1 - 2\beta_n - L^2\beta_n^2]\|x_n - Tx_n\|^2 \\
&\leq \|x_n - x^*\|^2 \qquad (3.13)
\end{aligned}$$

Furthermore, from (3.0) and (3.11) we get that

$$\|y_n - x_n\| = \beta_n\|Tx_n - x_n\| \leq \|Tx_n - x_n\| \to 0$$

as $n \to \infty$. Also, since $T$ is Lipschitz imply that,

$$\begin{aligned}
\|z_n - x_n\| &= \|\gamma_n(Ty_n - x_n) + \gamma_n(Tx_n - Tx_n)\| \\
&\leq \gamma_n L\|y_n - x_n\| + \gamma_n\|Tx_n - x_n\| \\
&\leq \gamma_n L\|Tx_n - x_n\| + \gamma_n\|Tx_n - x_n\| \to 0 \qquad (3.14)
\end{aligned}$$

Applying lemma 2.3.4 to (3.12), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|P_k[(1-\alpha_n)z_n+\alpha_n(z_n-x_n)]-x^*\|^2\\ &\leq \|(1-\alpha_n)(z_n-x^*)+\alpha_n(z_n-x_n)-\alpha_n x^*\|^2\\ &\leq \|(1-\alpha_n)(z_n-x^*)+\alpha_n(z_n-x_n)\|^2\\ &\quad -2\alpha_n\langle x^*, x_{n+1}-x^*\rangle\\ &= (1-\alpha_n)\|z_n-x^*\|^2+\alpha_n\|z_n-x_n\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|x_n-x^*\|^2-2\alpha_n\langle x^*, x_{n+1}-x^*\rangle\\ &\leq (1-\alpha_n)\|x_n-x^*\|^2+\|z_n-x_n\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|x_n-x^*\|^2-2\alpha_n\langle x^*, x_{n+1}-x^*\rangle\\ &\leq (1-\alpha_n)^2\|x_n-x^*\|^2+\|z_n-x_n\|^2-2\alpha_n\langle x^*, x_{n+1}-x^*\rangle\end{aligned}$$

It is clear that;

$$\|z_n-x_n\|^2\leq(\gamma_n L\|Tx_n-x_n\|+\gamma_n\|Tx_n-x_n\|)^2\to 0 \quad as \quad n\to\infty \quad and$$

$$\lim_{n\to\infty}\langle x^*, x_{n+1}-x^*\rangle=0$$

$$\begin{aligned}\Rightarrow \|x_{n+1}-x^*\|^2 &\leq (1-\alpha_n)\|x_n-x^*\|^2\\ &\quad -2\alpha_n\langle x^*, x_{n+1}-x^*\rangle\end{aligned}\tag{3.15}$$

$$Since \quad \lim_{n\to\infty}\langle x^*, x_{n+1}-x^*\rangle=0$$

$$\Rightarrow \|x_{n+1}-x^*\|^2\leq(1-\alpha_n)\|x_n-x^*\|^2$$

Applying lemma 2.1.8 to (3.15) we immediately deduce that $x_n\to x^*$.

**Case 2.** Assume that $\{\|x_n-p\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n=\|x_n-p\|^2$ and let $\tau:N\to N$ be a mapping for all $n\geq n_0$ (for some $n_0$ large enough) defined by

$$\tau(n)=max\{k\in N: k\leq n, \Gamma_k\leq\Gamma_{k+1}\}.$$

clearly, $\tau$ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$

From (3.10), it is easy to see that

$$\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 = \frac{M\alpha_{\tau(n)}}{\gamma_{\tau(n)}\beta_{\tau(n)}[1 - 2\beta_{\tau(n)} - L^2\beta^2_{\tau(n)}]} \to 0$$

Thus $\|x_{\tau(n)} - Tx_{\tau(n)}\| \to 0$.

By the same similar argument as above in case 1, we conclude immediately that $x_{\tau(n)}$ weakly converges to $x^*$ as $\tau(n) \to \infty$. At the same time we note that, for all $n_0 \geq n$,

$0 \leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2$

$\leq \alpha_{\tau(n)}[2\langle x^*, x^* - x_{\tau(n)+1}\rangle - \|x_{\tau(n)} - x^*\|^2]$

Hence, we have that

$\lim_{n\to\infty} \|x_{\tau(n)} - x^*\|^2 = 0$

therefore,

$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$

Furthermore, for $n \geq n_0$, it is easily observed that

$\Gamma_{\tau(n)} = \Gamma_{\tau(n)+1}$, if $n \neq \tau(n)$ (that is $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for

$\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$

$0 \leq \Gamma_{\tau(n)} \leq max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}$

Hence $\lim_{n\to\infty} \Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to $x^*$.

This complete the proof.

## A prototype example for our parameters is

$$\gamma_n = \frac{n^2 - 1}{4(n+1)^2(2 + L^2)}; \beta_n = \frac{n}{4(n+1)(2 + L^2)}; \alpha_n = \frac{1}{n+1}.$$

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