STRONG CONVERGENCE OF MODIFIED AVERAGING ITERATIVE ALGORITHM FOR ASYMPTOTICALLY NONEXPANSIVE MAPS

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CERTIFICATION

OGBUISI, FERDINARD UDOCHUKWU, a postgraduate student in the DEPARTMENT of MATHEMATICS with registration number PG/M.Sc./10/57024 has satisfactorily completed the requirements for research work for the degree MASTER of SCIENCE in MATHEMATICS. The work embodied in this project report is original and has not been submitted in part or full for any other diploma or degree of this or any other university.

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DEDICATION

To

OGBUISI, EVERISTUS OBIORAH

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Contents

ABSTRACT

Let H be a real Hilbert space and K a nonempty, closed and convex subset of H. Let $T : K \to K$ be an asymptotically nonexpansive map with a nonempty fixed points set. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be real sequences in (0,1). Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in K$ by

$$\begin{cases} y_n = P_K[(1-t_n)x_n], & n \geq 0 \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T^n y_n, & n \geq 0. \end{cases}$$

where $P_K : H \to K$ is the metric projection. Under some appropriate mild conditions on $\{\alpha_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$, we prove that $\{x_n\}$ converges strongly to fixed point of T. No compactness assumption is imposed on T and or K and no further requirement is imposed on the fixed point set $Fix(T)$ of T.

Chapter 1

Introduction

1.1 General Introduction

The theory of fixed points of nonlinear operators has found many powerful and important applications in diverse fields such as Differential Equations,Topology, Economics, Biology, Chemistry, Engineering, Game Theory, Physics, Dynamics, Optimal Control, and Functional Analysis . Iterative algorithms for approximating fixed points of some nonlinear operators belonging to certain classes of mappings that generalize nonexpansive mappings and defined in appropriate Banach spaces have been flourishing area of research for many mathematicians.The class of nonlinear mappings we studied in this work, is the class of asymptotically nonexpansive mappings. This class of asymptotically nonexpansive mappings which has engaged the interest of many researchers(for example see [18] and the references there in) was first introduced by Goebel and Kirk [10] in the year 1972.

Definition 1.1.1: Let K be a nonempty subset of a normed linear space E. A mapping $T : K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in K.$$

Definition 1.1.2: Let K be a nonempty subset of a normed linear space E. A mapping $T : K \to K$ is called asymptotically nonexpansive, if there exists a sequence $\{k_n\}, \ k_n \in$

$[1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

holds for each $x, y \in K$ and for each integer $n \geq 1$. It is clear that every nonexpansive mapping is asymptotically nonexpansive with $k_n = 1 \ \forall n \geq 1$.

The following example reveals that the class of asymptotically nonexpansive mappings properly contains the class of nonexpansive mappings.

Example 1 (Goebel and Kirk [10]). Let B denote the unit ball in the Hilbert space l^2 and let T be defined as follows:

$$T : (x_1, x_2, x_3, ...) \to (0, x_1^2, a_2 x_2, a_3 x_3, ...).$$

where $\{a_i\}$ is a sequence of numbers such that $0 < a_i < 1$ and $\prod_{i=2}^{\infty} a_i = \dfrac{1}{2}$.

Then, T is Lipschitz and $\|Tx - Ty\| \leq 2\|x - y\|, \ \forall x, y \in B$.

Moreover,

$$\|T^i x - T^i y\| \leq 2 \prod_{j=2}^{i} a_j \|x - y\| \ \forall i = 2, 3, ...$$

Thus,

$$\lim_{i\to\infty} k_i = \lim_{i\to\infty} 2 \prod_{j=2}^{i} a_j = 1$$

But let,

$$x = (\frac{2}{3}, 0, 0, ...)$$

and

$$y = (\frac{1}{2}, 0, 0, ...),$$

clearly x, y are in B.

$$\|Tx - Ty\| = |\frac{4}{9} - \frac{1}{4}| = \frac{7}{36} > \frac{1}{6} = \|x - y\|.$$

therefore T is not nonexpasnsive.

1.2 Some Banach Spaces and their Properties.

In this section we give the definition of some special Banach spaces as well as some of their geometric properties.

Definition 1.2.1 A Banach space E is said to be uniformly convex if for any $\epsilon \in (0, 2]$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in E$ with $||x|| \leq 1$, $||y|| \leq 1$ and $||x-y|| > \epsilon$, then $||\frac{1}{2}(x + y)|| \leq 1 - \delta$. Geometrically, a Banach space is uniformly convex if the unit ball centred at the origin is uniformly round. Again, uniform convexity is a property of the norm on E.

The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf \{1 - ||\frac{x+y}{2}|| : ||x|| = 1 = ||y||, ||x - y|| \geq \epsilon\},$$

$0 < \epsilon \leq 2$.

Definition 1.2.2 A Banach space E is said to be smooth if for all $x \in E$, $x \neq 0$ with $||x|| = 1$, there exists $f \in E^*$ such that $\langle x, f\rangle = 1$. Smoothness is a property of the norm. In fact, E is smooth if and only if $\forall x, y \in E$, $x \neq 0$

$$\lim_{t\to 0} \frac{||x + ty|| - ||x||}{t} \tag{1.1}$$

exists. The limit, when it exits, is of the form $f_x(y)$ with $f_x \in E^*$ and is called the Gáteaux derivative of the norm in E. Thus T is smooth if and only if the norm is Gáteaux differentiable.

Definition 1.2.3 A Banach space is uniformly smooth if and only if the limit (1.1) exists uniformly on the set

$$U = \{(x, y) \in E \times E : ||x|| = ||y|| = 1\}.$$

Definition 1.2.4 A Banach space E is said to be an Opial space (see for example [1], [16]) if for each sequence $\{x_n\}_{n=1}^{\infty}$ in E which converges weakly to a point $x \in E$

$$\liminf ||x_n - x|| < \liminf ||x_n - y||,$$

for all $y \in E, y \neq x$. It is known (see [17]) that every Hilbert space and every $l_p(1 < p < \infty)$ space enjoy the property. Also in [9], D Van showed that any separable Banach space can be equivalently re-normed so that it satisfies Opial condition. Indeed, for any normed space E the existence of a weakly sequentially continuous duality map implies that E is an Opial's space, but the converse implication does not hold. Notably, the Lebesgue space L_p is not an Opial space for $p \neq 2$.

Definition 1.2.5 A function $\Phi : [0,\infty) \to [0,\infty)$ is said to be a guage function if Φ is continuous and strictly increasing with $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$.

Definition 1.2.6 Let E be a real Banach space. Let E^* denote the topological dual of E and 2^{E^*} be the collection of all subsets of E^*. Let $\Phi : [0,\infty) \to [0,\infty)$ be a gauge function. The mapping J_Φ:$E \to 2^{E^*}$ defined by

$$J_\Phi x = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \Phi(\|x\|)\}.$$

is called duality map with gauge function Φ, where $\langle ., .\rangle$ denotes the generalized duality paring between E and E^*. We note that if $1 < q < \infty$, then $\Phi(t) = t^{q-1}$ is a gauge function. The duality mapping $J_q : E \to 2^{E^*}$ with gauge $\Phi(t) = t^{q-1}$ defined for each $x \in E$ by

$$J_\Phi x = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|^{q-1}\}.$$

is called the *generalised duality mapping.* If $q = 2$, we obtain

$$J_2 := J : E \to 2^{E^*},$$

defined for all $x \in E$ by

$$J_2 x := J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

J_2 is known as the normalised duality map. It is well known (see for instance [2, 8]) that for $1 < q < \infty, \quad J_q(x) = \|x\|^{q-2}J(x), for\ x \in X, x \neq 0$. The following theorem has been proved for uniformly convex Banach space.

Theorem 1.2.7(Xu, [30]) Let $p > 1$ and $r > 0$ be two fixed real numbers.Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing and convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+, \quad g(0) = 0,$$

such that for all $x, y \in B_r$ and $0 \leq \lambda \leq 1$,

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|). \tag{1.3}$$

where $W_p(\lambda) := \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$ and $B_r := \{x \in X : \|x\| \leq r\}$.

The next result below establishes the existence of a fixed point for asymptotically nonexpansive mapping in a nonempty, closed, convex and bounded subset of a uniformly convex Banach space.

Theorem 1.2.8([10] Theorem 1) Let K be a nonemtpty, closed, convex and bounded subset of a uniformly convex Banach space X, let $F : K \to K$ be an asymptotically nonexpansive mapping,then F has a fixed point.

Proof For each $x \in K$ and $r > 0$, Let $S(x, r)$ denote the spherical ball centred at x with radius r. Let $y \in K$ be fixed, and let the set R_y consist of those numbers ρ for which there exists an integer k such that

$$K \cap (\cap_{i=k}^{\infty} S(F^i y, \rho)) \neq \emptyset.$$

If d is the diameter of K then $d \in R_y$, so $R_y \neq \emptyset$. Let $\rho_0 = g.l.b.R_y$, and for each $\epsilon > 0$, define $C_\epsilon = \cup_{k=1}^{\infty}(\cap_{i=k}^{\infty} S(F^i y, \rho + \epsilon))$. Thus for each $\epsilon > 0$ the set $C_\epsilon \cap K$ are nonempty and convex, so reflexivity of X implies that

$$C = \cap_{\epsilon > 0}(\bar{C}_\epsilon \cap K) \neq \emptyset.$$

Note that for $x \in C$ and $\eta > 0$ there exists an integer N such that if $i \geq N$, $\|x - F^i y\| \leq \rho_0 + \eta$.

Now let $x \in C$ and suppose the sequence $\{F^n x\}$ does not converge to x (i.e., suppose $Fx \neq x$). Then there exists $\epsilon > 0$ and a subsequence $\{F^{n_i} x\}$ of $\{F^n x\}$ such that

$$\|F^{n_i} x - x\| \geq \epsilon, \ i = 1, 2, ...$$

For $m > n$,

$$\|F^n x - F^m x\| \leq k_n \|x - F^{m-n} x\|,$$

where k_n is the Lipschitz constant for F^n obtained from the definition of asymptotic nonexpansiveness. Assume $\rho_0 > 0$ and choose $\alpha > 0$ so that $(1 - \delta(\frac{\epsilon}{\rho_0 + \alpha}))(\rho_0 + \alpha) < \rho_0$. Select n so that $\|x - F^n x\| \geq \epsilon$ and also that $k_n(\rho_0 + \dfrac{\alpha}{2}) \leq \rho_0 + \alpha$. If $N \geq n$ is sufficiently large, then $m > N$ implies

$$\|x - F^{m-n} y\| \leq \rho_0 + \frac{\alpha}{2}$$

and we have

$$\|F^n x - F^m y\| \leq k_n \|x - F^{m-n} y\| \leq \rho_0 + \alpha,$$

$$\|x - F^m y\| \leq \rho_0 + \alpha.$$

Thus by uniform convexity of X, if $m > n$,

$$\|(\frac{x + F^n x}{2}) - F^m y\| \leq (1 - \delta(\frac{\epsilon}{\rho_0 + \alpha}))(\rho_0 + \alpha) < \rho_0,$$

and this contradicts the definition of ρ_0. Hence we conclude $\rho_0 = 0$ or $Fx = x$. But $\rho_0 = 0$ implies $\{F^n y\}$ is a Cauchy sequence yielding $F^n y \to x = Fx$ as $n \to \infty$. Therefore the set consists of a single point which is a fixed point under F.

Kirk, Yanez and Shin [13] improved the result of Goebel and Kirk [10]. They proved that if a reflexive Banach space E has the property that each of its closed, bounded and convex subset has the fixed point property for nonexpansive maps, then it will also have the fixed point property for asymptotically nonexpansive mapping which has a nonexpansive iterate.

Theorem 1.2.9 Let H be a real Hilbert space and C a nonempty closed and convex subset of H. Let $T : C \to C$ be an asymptotically non expansive map. Then $Fix(T) = \{x \in C : Tx = x\}$ is closed and convex.

Proof:

Convexity.

For any $x, y \in Fix(T)$ and $\alpha \in (0,1)$. Let $z_\alpha := (1-\alpha)x + \alpha y$. Then,

$$\begin{aligned}
\|T^n z_\alpha - z_\alpha\|^2 &= \|T^n z_\alpha - [(1-\alpha)x + \alpha y]\|^2 \\
&= \|(1-\alpha)(T^n z_\alpha - x) + \alpha(T^n z_\alpha - y)\|^2 \\
&= (1-\alpha)\|T^n z_\alpha - x\|^2 + \alpha\|T^n z_\alpha - y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&\leq (1-\alpha)k_n^2\|z_\alpha - x\|^2 + \alpha k_n^2\|z_\alpha - y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= (1-\alpha)k_n^2\|(1-\alpha)x + \alpha y - x\|^2 + \alpha k_n^2\|(1-\alpha)x + \alpha y - y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha^2(1-\alpha)k_n^2\|x-y\|^2 + \alpha k_n^2\|(1-\alpha)x + (1-\alpha)y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha^2(1-\alpha)k_n^2\|x-y\|^2 + \alpha(1-\alpha)^2k_n^2\|x-y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha(1-\alpha)[\alpha k_n^2 + (1-\alpha)k_n^2 - 1]\|x-y\|^2 \\
&= \alpha(1-\alpha)(k_n^2 - 1)\|x-y\|^2 \to 0.
\end{aligned}$$

Therefore,

$$\|T^n z_\alpha - z_\alpha\| \to 0.$$

Now,

$$\begin{aligned}
0 &\leq \|Tz_\alpha - z_\alpha\| \\
&\leq \|Tz_\alpha - T^n z_\alpha\| + \|T^n z_\alpha - z_\alpha\| \\
&\leq k_1\|z_\alpha - T^{n-1}z_\alpha\| + \|T^n z_\alpha - z_\alpha\| \to 0.
\end{aligned}$$

Hence

$$Tz_\alpha = z_\alpha.$$

We now show that $Fix(T)$ is closed. Let $\{x_n\} \subseteq Fix(T)$ be arbitrary and let $x_n \to x$ as $n \to \infty$, we show that x is in $Fix(T)$.

$$Tx = T\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_n = x.$$

Two other definitions of asymptotically nonexpansive maps has also appeared in the literature. One of the definitions which is weaker than Definition 1.1.2 was introduced by Kirk[12] and requires that

$$\limsup_{n\to\infty}\sup_{y\in K}(\|T^n x - T^n y\| - \|x-y\|) \leq 0.$$

for every $x \in K$ and that T^N be continuous for some integer $N > 1$.

The other definition which has appeared require that

$$\limsup_{n\to\infty}(\|T^n x - T^n y\| - \|x-y\|) \leq 0 \ \forall x, y \in K.$$

This, however, has been shown to be unsatisfactory from the point of view of fixed point theory. Tingly [24] constructed an example of a closed convex set K in a Hilbert space and a continuous map $T : K \to K$ which actually satisfies the following condition $\lim_{n\to\infty}\|T^n x - T^n y\| = 0 \ \forall x, y \in K$ but has no fixed point.

1.3 Iterative Algorithms for Asymptotically Nonexpansive Mappings

1.3.1 Modified Mann Iterative Algorithm

The averaging iteration process,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1,$$

where $T : K \to K$ is asymptotically nonexpansive in the sense of definition 1.1.2, K a closed, convex and bounded subset of a Hilbert space was introduced by Schu[23].

In [21] Schu used the modified Mann iteration method,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, n \geq 1$$

and proved the following theorem.

Theorem(1.3.1): Let H be a Hilbert space, K a nonempty closed convex and bounded subset of H. Let $T : K \to K$ be a completely continuous asymptotically nonexpansive map with sequence $\{k_n\}_{n=1}^{\infty}$ with $k_n \in [1, \infty)$ for all $n \geq 1, \lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in [0,1] satifying the condition $\epsilon < \alpha_n < 1 - \epsilon$ for some $\epsilon > 0$. Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, n \geq 1,$$

converges strongly to a fixed point of T.

1.3.2 Iterative method of Schu

In this subsection, consider algorithm for approximating fixed points of asymptotically nonexpansive mappings which deals with almost fixed points,

$$x_n := \mu_n T^n x_n$$

of an asymptotically nonexpansive mappings T. Schu [24] proved the convergence of this sequence $\{x_n\}$ to some fixed point of T under additional assumption that T is uniformly asymptotically regular and $(I - T)$ is demiclosed. These assumptions had actually been made by Vijayaraju[27] to ensure the existence of a fixed point of T By strengthening the asymptotic regularity of T, Schu established the convergence of an explicit iteration scheme,

$$z_{n+1} := \mu_n T^n z_n$$

to some fixed point of T.

1.3.3 Halpern-type process

One of the most useful results concerning algorithms for approximating fixed points of non-expansive mappings in real uniformly smooth Banach spaces is the celebrated convergence theorem of Riech[29] who proved that the implicit sequence $\{x_n\}$ defined as,

$$x_n = \frac{1}{n}u + (1 - \frac{1}{n})Tx_n.$$

converges strongly to a fixed point of T. Several authors have tried to obtain a result analogous to that of Reich [19] for asymptotically nonexpansive mappings. Suppose K is a nonempty bounded closed convex subset of a real uniformly smooth Banach space E and $T : K \to K$ is an asymptotically nonexpansive mapping with sequence $k_n \geq 1$ for all $n \geq 1$. Fix $u \in K$ and define, for each integer $n \geq 1$, the contraction mapping $S_n : K \to K$ by, (see [6]),

$$S_n(x) = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x,$$

where $\{t_n\} \subset [0, 1)$ is any sequence such that $t_n \to 1$.Then by the Banach Contraction Mapping Principle, there is a unique point x_n fixed by S_n, i.e. there is x_n such that

$$x_n = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x_n.$$

For some existing results on asymptotically nonexpansive maps, an interested reader should see [4,7,14,18,20,22] and the references there in.

1.4 Organization of Thesis

We have introduced in this chapter (Chapter One), various iteration methods for asymptotically nonexpansive maps and some existing results on them. We also studied some of the important Banach spaces which are encountered in this work and their properties.
In Chapter Two (the Preliminaries), we presented most of the classical results on the sequences of real numbers encountered in Operator Theory. We also looked at projection maps and some other results vital to this work.
In Chapter Three, we study certain averaging iterative algorithm for approximating the fixed point of asymptotically nonexpansive mappings introduced by Goebel and Kirk [10].

Chapter 2

Preliminaries

2.1 Some Classical Results on Sequences of Real Numbers

We now state and prove the following important results concerning sequences of real numbers.

Lemma 2.1.1 Let $\{a_n\}_{n=1}^{\infty}$ be a real sequence of real numbers with $a_n \geq 0 \ \forall \ n$. Let $\liminf a_n = \delta > 0$. Then there exists n_0 such that $a_n \geq \frac{\delta}{2} \forall n \geq n_0$. Consequently there exists a subsequence $\{a_{n_j}\}_{n=1}^{\infty} \subset \{a_n\}_{n=1}^{\infty}$ such that $a_{n_j} \to \delta$ as $j \to \infty$.

Proof: Define $M_n := \inf_{j \geq n} a_j$, then $M_n \leq a_n \forall n$. Now,

$$\delta = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} (\inf_{j\geq n} a_j) = \lim_{n\to\infty} M_n.$$

Hence for $\epsilon = \frac{\delta}{2} > 0$, $\exists \ \ n_0$ such that

$$|M_n - \delta| \leq \frac{\delta}{2} \forall n \geq n_0.$$

This implies that

$$\frac{\delta}{2} \leq M_n \leq a_n, \forall n \geq n_0.$$

Consequently, $\{M_n\}_{n=1}^{\infty} \subset \{a_n\}_{n=1}^{\infty}$ and

$$\delta = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} (\inf_{j\geq n} a_j) = \lim_{n\to\infty} M_n.$$

An important application of lemma 2.1.1 can be seen in the following lemma.

Lemma 2.1.2 Suppose $a_n, \alpha_n \geq 0 \; \forall \; n$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$, $\sum\limits_{n=1}^{\infty} \alpha_n a_n < \infty$. Then $\liminf\limits_{n\to\infty} a_n = 0$.

Proof: Suppose that $\liminf\limits_{n\to\infty} a_n = \delta > 0$, then by Lemma 2.1.1 there exists n_0 such that $a_n \geq \frac{\delta}{4} \; \forall \; n \geq n_0$. Hence,

$$\alpha_n a_n > \alpha_n \frac{\delta}{4}, \forall \; n \geq n_0.$$

This implies that,

$$\sum_{n=1}^{\infty} \alpha_n a_n > \sum_{n=1}^{\infty} \frac{\delta}{4} \alpha_n = \frac{\delta}{4} \sum_{n=1}^{\infty} \alpha_n = \infty.$$

This contradiction implies that $\liminf\limits_{n\to\infty} a_n = 0$.

Lemma 2.1.3 Let $a_n, b_n \geq 0$, $\forall n$. Let $\alpha \in [0, 1)$, and $\lim\limits_{n\to\infty} b_n = 0$. If $a_{n+1} \leq \alpha a_n + b_n$, then $\lim\limits_{n\to\infty} a_n = 0$

Proof:

$$a_{n+1} \leq \alpha a_n + b_n = a_n - (1 - \alpha) a_n + b_n.$$

Claim : $\liminf\limits_{n\to\infty} a_n = 0$.

Proof: Suppose $\liminf\limits_{n\to\infty} a_n = \delta > 0$. By Lemma 2.1.1 $\exists \; n_0$ such that

$$a_n \geq \frac{\delta}{2}, \forall \; n \geq n_0.$$

Also since $\lim\limits_{n\to\infty} b_n = 0$, $\exists \; n_1$ such that

$$b_n \leq \frac{\delta}{4}(1 - \alpha), \forall n.$$

Hence,for $N = \max\{n_0, n_1\}$ we have

$$\begin{aligned}
a_{n+1} &\leq a_n - (1-\alpha)\frac{\delta}{2} + \frac{\delta}{4}(1-\alpha), \forall n > N \\
&= a_n - \frac{\delta}{4}(1-\alpha) \\
&\Rightarrow \frac{\delta}{4}(1-\alpha) \leq a_n - a_{n+1} \\
&\Rightarrow \sum_{j=1}^{n} \frac{\delta}{4}(1-\alpha) \leq \sum_{j=1}^{n} a_j - a_{j+1} = a_1 - a_{n+1} \leq a_1 \\
&\Rightarrow n[\frac{\delta}{4}(1-\alpha)] \leq a_1 \ \forall \ n \geq 1 \\
&\Rightarrow \lim_{n\to\infty} n[\frac{\delta}{4}(1-\alpha)] \leq \lim_{n\to\infty} a_1 = a_1 \\
&\Rightarrow \infty \leq a_1(\text{contradiction}).
\end{aligned}$$

Hence $\liminf_{n\to\infty} a_n = 0$. From Lemma 2.1.1, there exists a subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to 0$ as $k \to \infty$. Also, $\lim_{n\to\infty} b_n = 0$ implies that $b_{n_k} \to 0$ as $k \to \infty$ for all subsequences $\{b_{n_k}\}_{k=1}^{\infty} \subset \{b_n\}_{n=1}^{\infty}$. Therefore there exists k_0 such that $a_{n_k} < \epsilon$, $b_{n_k} < (1-\alpha)\epsilon$ for all $k \geq k_0$. Hence $a_{n_{k_0}} < \epsilon$, $b_{n_{k_0}} < (1-\alpha)\epsilon$. We now show that,

$$a_{n_{k_0}+m} < \epsilon, \forall m \geq 1.$$

We do this by induction. observe that $b_{n_{k_0}+m} < (1-\alpha)\epsilon$. for all $m \geq 1$.Now for $m = 1$,

$$\begin{aligned}
a_{n_{k_0}+1} &\leq \alpha a_{n_{k_0}} + b_{n_{k_0}} \\
&< \alpha\epsilon + (1-\alpha)\epsilon = \epsilon.
\end{aligned}$$

We assume it is true for $m = n$, i.e., $a_{n_{k_0}+n} < \epsilon$ and prove for $m = n+1$. Now,

$$\begin{aligned}
a_{n_{k_0}+n+1} &\leq \alpha a_{n_{k_0}+n} + b_{n_{k_0}+n} \\
&< \alpha\epsilon + (1-\alpha)\epsilon = \epsilon.
\end{aligned}$$

Hence $a_{n_{k_0}+m} < \epsilon$ for all $m \geq 1$ so that $\lim_{n\to\infty} a_n = 0$.

Immediate consequences of Lemma 2.1.3 are the following:

Corollary 2.1.1 Let $a_n, b_n \geq 0 \forall\ n$, $\sum_{n=1}^{\infty} b_n < \infty$. Let $\alpha \in [0,1)$, If $a_{n+1} \leq \alpha a_n + b_n$, then $\lim_{n\to\infty} a_n = 0$.

Proof: Obvious from Lemma 2.1.3.

Alitre: The above result can be proved directly as follows:

$$a_{n+1} \leq a_n - (1-\alpha)a_n + b_n$$

$$\Rightarrow (1-\alpha)a_n \leq a_n - a_{n+1} + b_n$$
$$\Rightarrow (1-\alpha)\sum_{j=1}^{n} a_j \leq \sum_{j=1}^{n}(a_j - a_{j+1} + b_j)$$
$$\Rightarrow (1-\alpha)\sum_{j=1}^{n} a_j \leq a_1 - a_{n+1} + \sum_{j=1}^{n} b_j$$
$$\Rightarrow (1-\alpha)\lim_{n\to\infty}\sum_{j=1}^{n} a_j \leq a_1 + \lim_{n\to\infty}\sum_{j=1}^{n} b_j$$
$$\Rightarrow \sum_{j=1}^{\infty} a_j \leq \infty, \qquad \Rightarrow a_j \to 0.$$

Corollary 2.1.2 Let $a_n \geq 0 \; \forall \; n$, $\alpha \in [0,1)$. If $a_{n+1} \leq \alpha a_n \forall n \geq 1$, then $\lim_{n\to\infty} a_n = 0$.

Remark 2.1.1 If we replace $\alpha \in [0,1)$ with $\{\alpha_n\} \subset [0,1)$, in Lemma 2.1.3 and Corollary 2.1.1, that $a_{n+1} \leq \alpha_n a_n + b_n$,and $\lim_{n\to\infty} a_n$ may not be zero can be seen with the following examples.

Example 2.1.1: Take $a_n = \frac{n+1}{n}$, $b_n = \frac{2}{n+1}$, $\alpha_n = \frac{n}{n+1}$, $\forall n \geq 1$. Then $a_{n+1} \leq \alpha_n a_n + b_n$, $\lim_{n\to\infty} b_n = 0$ but $\lim_{n\to\infty} a_n \neq 0$.

Example 2.1.2: Take $a_n = \frac{n}{n+1}$, $b_n = \frac{2}{n(n+2)}$, $\alpha_n = \frac{(n^2-1)}{n^2}$, $\forall n \geq 1$. Then $a_{n+1} \leq \alpha_n a_n + b_n$, $\sum_{n=1}^{\infty} b_n < \infty$ but $\lim_{n\to\infty} a_n \neq 0$.

Lemma 2.1.4 Let $\Phi : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function with $\Phi(0) = 0$. If $\sum\limits_{n=1}^{\infty} \Phi(a_n) < \infty$ then $\lim_{n\to\infty} a_n = 0$.

Proof: Let $\sum\limits_{n=1}^{\infty} \Phi(a_n) < \infty$, then $\lim_{n\to\infty} \Phi(a_n) = 0$. Also, since Φ is continuous we have

$$\lim_{n\to\infty} \Phi(a_n) = \Phi(\lim_{n\to\infty} a_n) = 0.$$

So that

$$\Phi^{-1}(\Phi \lim_{n\to\infty} a_n) = \Phi^{-1}(0) = 0.$$

Hence, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.1.5: Let $\Phi : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function. Let $\{\alpha_n\} \subseteq [0,1)$ be such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $\{a_n\}$ is such that $a_n \geq 0\ \forall\ \ n \geq\ \ 0$ with $\sum_{n=1}^{\infty} \alpha_n \Phi(a_n) < \infty$, then $\liminf_{n\to\infty} a_n = 0$.

Proof: Assume for contradiction that $\liminf_{n\to\infty} a_n = \delta > 0$, then by Lemma 2.1.1, there exists n_0 such that $a_n \geq \frac{\delta}{2}\ \forall \geq n_0$. Since Φ is strictly increasing, $\Phi(a_n) \geq \Phi(\frac{\delta}{2})$ and $\alpha_n\Phi(a_n) \geq \alpha_n\Phi(\frac{\delta}{2})\ \forall n \geq n_0$. Hence

$$\sum_{n=1}^{\infty} \alpha_n \Phi(a_n) \geq \sum_{n=1}^{\infty} \alpha_n \Phi(\frac{\delta}{2}) = \infty.$$

The above contradiction implies that $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.1.6: Let $a_n, b_n \geq 0\ \forall\ n$, $\sum_{n=1}^{\infty} b_n < \infty$. If $a_{n+1} \leq a_n + b_n$, Then $\lim_{n\to\infty} a_n$ exists. Consequently, if $\liminf_{n\to\infty} a_n =\ 0$ then $\lim_{n\to\infty} a_n = 0$.

Proof:

$$\begin{aligned}
a_{n+m} &\leq a_{n+m-1} + b_{n+m-1} \\
&\leq a_{n+m-2} + b_{n+m-2} + b_{n+m-1} \\
&\leq a_{n+m-3} + b_{n+m-3} + b_{n+m-2} + b_{n+m-1} \\
&\ . \\
&\ . \\
&\ . \\
&\leq a_n + \sum_{j=n}^{n+m} b_j
\end{aligned}$$

Now,

$$\liminf_{n\to\infty} a_n = \liminf_{m\to\infty} a_m = \liminf_{m\to\infty} a_{m+n} \leq \limsup_{m\to\infty} a_{n+m} \leq a_n + \sum_{j=n}^{\infty} b_j.$$

Hence,

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n = \limsup_{m\to\infty} a_m \leq \liminf_{n\to\infty} a_n.$$

Therefore, $\lim_{n\to\infty} a_n$ exists. Consequently, if $\liminf a_n = 0$, $\lim a_n = 0$ since $\lim a_n = \liminf a_n$.

Lemma 2.1.7: Let a_n, $b_n \geq 0\ \forall n$, $\{\delta_n\} \subset [0,1)$, $\sum \delta_n = \infty$, $\sum b_n < \infty$. If $a_{n+1} \leq [1-\delta_n]a_n + b_n$, then $\lim a_n = 0$

Proof: Since $a_{n+1} \leq a_n - \delta_n a_n + b_n$, we obtain

$$\begin{aligned}
&\Rightarrow \quad \delta_n a_n \leq a_n - a_{n+1} + b_n \\
&\Rightarrow \quad \sum_{j=1}^{n} \delta_j a_j \leq \sum_{j=1}^{n} a_j - a_{j+1} + b_j \\
&\Rightarrow \quad \sum_{j=1}^{n} \delta_j a_j \leq a_1 - a_{n+1} + \sum_{j=1}^{n} b_j \leq M.
\end{aligned}$$

Hence $\sum_{n=1}^{\infty} \delta_n a_n < \infty$ so that $\liminf a_n = 0$ by Lemma 2.1.2.

By Lemma 2.1.1, there exists $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to 0$ as $k \to \infty$. Also, $\sum_{n=1}^{\infty} b_n < \infty$ implies that $\lim_{n\to\infty} b_n = 0$ so that $b_{n_k} \to 0$ for all subsequences $\{b_{n_k}\} \subset \{b_n\}$. It then follows that for every $\epsilon > 0$ there exists k_1, k_2 such that $a_{n_k} < \epsilon\ \forall k \geq k_1$ and $b_{n_k} < (\delta_{n_{k_2}})\epsilon$ for all $k \geq k_2$. Let $k_0 = max\{k_1, k_2\}$ then $a_{n_k} < \epsilon\ \forall \geq k_0$ and $b_{n_{k_0}} < (\delta_{n_{k_0}})\epsilon$ $\forall k \geq k_0$. We then have

$$a_{n_{k_0}} < \epsilon,$$

and

$$b_{n_{k_0}} < (\delta_{n_{k_0}})\epsilon.$$

Also, $b_{n_{k_0}+m} < (\delta_{n_{k_0}+m})\epsilon\ \forall m$. We now show that

$$a_{n_{k_0}+m} < \epsilon, \forall m.$$

We do this by mathematical induction. For $m = 1$,

$$\begin{aligned}
a_{n_{k_0}+1} &\leq a_{n_{k_0}}[1-\delta_{n_{k_0}}] + b_{n_{k_0}} \\
&< \epsilon[1-\delta_{n_{k_0}}] + \delta_{n_{k_0}}\epsilon \\
&= \epsilon
\end{aligned}$$

For $m = 2$,

$$\begin{aligned} a_{n_{k_0}+2} &\leq a_{n_{k_0}+1}[1-\delta_{n_{k_0}+1}] + b_{n_{k_0}+1} \\ &< \epsilon[1-\delta_{n_{k_0}+1}] + \delta_{n_{k_0}+1}\epsilon \\ &= \epsilon \end{aligned}$$

We assume it is true for $m = n$ that is $a_{n_{k_o}+n} < \epsilon$ and prove for $m = n+1$.

$$\begin{aligned} a_{n_{k_0}+n+1} &\leq a_{n_{k_0}+n}[1-\delta_{n_{k_0}+n}] + b_{n_{k_0}+n} \\ &< \epsilon[1-\delta_{n_{k_0}+n}] + \delta_{n_{k_0}+n}\epsilon \\ &= \epsilon \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} a_n = 0$.

Corollary 2.1.3: Suppose in Lemma 2.1.7 $\sum\limits_{n=1}^{\infty} b_n < \infty$ is replaced with $\lim\limits_{n\to\infty} \frac{b_n}{\delta_n} = 0$, the result still holds.

Proof: Since $a_{n+1} \leq [1-\delta_n]a_n + b_n$ can be written as

$$a_{n+1} \leq a_n - \delta_n a_n + \frac{b_n}{\delta_n}\delta_n,$$

we claim that $\liminf\limits_{n\to\infty} a_n = 0$, for suppose that $\liminf\limits_{n\to\infty} a_n = \delta > 0$, then there exists n_1 such that $a_n \geq \frac{\delta}{2}$ for all $n \geq n_1$ so that $\delta_n a_n \geq \delta_n \frac{\delta}{2}$. Also since $\frac{b_n}{\delta_n} \to 0$ as $n \to \infty$ there exists n_2 such that $\frac{b_n}{\delta_n} \leq \frac{\delta}{4}$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$ then

$$a_{n+1} \leq a_n - \frac{\delta}{2}\delta_n + \frac{\delta}{4}\delta_n, \forall n \geq n_0.$$

Hence,

$$\frac{\delta}{4}\delta_n \leq a_n - a_{n+1},$$

$$\frac{\delta}{4}\sum_{j=n_0}^{m} \delta_j \leq \sum_{j=n_0}^{m} a_j - a_{j+1} \leq a_{n_0}.$$

Taking the limits as $m \to \infty$ of both sides of the inequality we have

$$\frac{\delta}{4}\sum_{n=n_0}^{\infty} \delta_n \leq a_{n_0}$$

The above contradiction shows that $\liminf_{n\to\infty} a_n = 0$. It then follows that there exists $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to 0$ as $n \to \infty$. Consequently, for every $\epsilon > 0$ there exists k_1 such that $a_{n_k} \le \frac{\epsilon}{2}$ for all $k \ge k_1$. Also since $\frac{b_n}{\delta_n} \to 0$ as $n \to \infty$ and δ_n is bounded we have that $\delta_n \frac{b_n}{\delta_n} \equiv \gamma_n \to 0$ as $n \to \infty$ so that $\gamma_{n_k} \to 0$ as $k \to \infty$ for all subsequences $\{\gamma_{n_k}\} \subset \{\gamma_n\}$. Hence, there exists k_2 such that $\gamma_{n_k} \le \frac{\epsilon}{2}$ for all $k \ge k_2$. Let $k_0 = \max\{k_1, k_2\}$ then $a_{n_{k_0}}, \gamma_{n_{k_0}} \le \frac{\epsilon}{2}$. It is now easy to show that

$$a_{n_{k_0}+m} \le \epsilon, \forall m \ge 1$$

using mathematical induction since

$$a_{n+1} \le a_n - \delta_n a_n + \gamma_n \le a_n + \gamma_n.$$

Lemma 2.1.8. Let $a_n, b_n \ge 0$, $\alpha_n \in [0,1)$ for all n, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} b_n < \infty$. Let $\Phi : [0,\infty) \to [0,\infty)$ be strictly increasing and continuous with $\Phi(0) = 0$. Let $a_{n+1} \le a_n - \alpha_n \Phi(a_n) + b_n$. Then $\lim_{n\to\infty} a_n = 0$.

Proof:

$$a_{n+1} \le a_n - \alpha_n \Phi(a_n) + b_n.$$

This implies that

$$\alpha_n \Phi(a_n) \le a_n - a_{n+1} + b_n,$$

so that

$$\sum_{j=1}^{n} \alpha_j \Phi(a_j) \le a_1 - a_{n+1} + \sum_{j=1}^{n} b_j < M.$$

Hence, $\sum_{n=1}^{\infty} \alpha_n \Phi(a_n) < \infty$ so that $\liminf_{n\to\infty} \Phi(a_n) = 0$ by lemma 2.1.1.

Claim: $\liminf_{n\to\infty} a_n = 0$

Proof: Suppose that $\liminf_{n\to\infty} a_n = \delta > 0$, by Lemma 2.1.1 $\exists$ n_0 such that $a_n \ge \frac{\delta}{2}$ for all $n \ge n_0$. Since Φ is strictly increasing then

$$\Phi(a_n) \ge \Phi(\frac{\delta}{2}) > \Phi(0) = 0.$$

which yields,

$$\liminf_{n\to\infty} \Phi(a_n) > 0.$$

This contradiction implies that $\liminf_{n\to\infty} a_n = 0$. It then follows from lemma 2.1.1 that there exists a subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to 0$ as $k \to \infty$. Consequently for every $\epsilon > 0$ there exists k_1, k_2, such that $a_{n_k} < \epsilon \forall k \geq k_1$, $\sum\limits_{j=n_k}^{\infty} b_j < \epsilon$ for all $k \geq k_2$. Let $k_0 = \max\{k_1, k_2,\}$ then $a_{n_k} < \epsilon$, $\sum\limits_{j=n_k}^{\infty} b_j < \alpha_{n_k}\Phi(a_{n_k})$ for all $k \geq k_0$. Hence, $a_{n_{k_0}} < \epsilon$. We now show that

$$a_{n_{k_0}+m} < \epsilon, \forall m \geq 1.$$

We do this by mathematical induction.

$$\begin{aligned}
a_{n_{k_0}+m} &\leq a_{n_{k_0}+m-1} - \alpha_{n_{k_0}+m-1}\Phi(a_{n_{k_0}+m-1}) + b_{n_{k_0}+m-1} \\
&\leq a_{n_{k_0}+m-2} - \alpha_{n_{k_0}+m-2}\Phi(a_{n_{k_0}+m-2}) + b_{n_{k_0}+m-2} \\
&. \\
&. \\
&. \\
&\leq a_{n_{k_0}} - \alpha_{n_{k_0}}\Phi(a_{n_{k_0}}) + \sum_{j=n_{k_0}}^{n_{k_0}+m-1} b_j \\
&< \epsilon.
\end{aligned}$$

Lemma 2.1.9 If $\sum\limits_{n=1}^{\infty} b_n < \infty$ is replaced with $\frac{b_n}{\alpha_n} \to 0$ as $n \to \infty$ in Lemma 2.1.8, the result still holds.

Proof: Write $b_n = \alpha_n \frac{b_n}{\alpha_n} = \alpha_n \beta_n$

Claim: $\liminf_{n\to\infty} a_n = 0$.

Proof: Suppose $\liminf_{n\to\infty} a_n = \delta > 0$, there

exists n_0 such that $a_n \geq \frac{\delta}{2}$, $\beta_n \leq \frac{1}{2}\Phi(\frac{\delta}{2})$ for all $n \geq n_0$. Hence,

$$\Phi(a_n) \geq \Phi(\frac{\delta}{2}), \forall\, n \geq n_0$$

so that

$$\begin{aligned}
a_{n+1} &\leq a_n - \alpha_n \Phi(\frac{\delta}{2}) + \frac{1}{2}\Phi(\frac{\delta}{2})\alpha_n \\
\alpha_n[\frac{1}{2}\Phi(\frac{\delta}{2})] &\leq a_n - a_{n+1} \\
\lim_{n\to\infty}\sum_{j=1}^{n}[\frac{1}{2}\Phi(\frac{\delta}{2})]\alpha_j &\leq a_1 \\
\text{Thus } \sum_{n=1}^{\infty}\frac{1}{2}\Phi(\frac{\delta}{2})\alpha_n &< \infty
\end{aligned}$$

The above contradiction implies that $\liminf a_n = 0$. Consequently, there exists $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to 0$ as $k \to \infty$. That is for every $\epsilon > 0$ there exists k_0 such that $a_{n_k} < \epsilon$ for all $k \geq k_0$ It then follows that $a_{n_{k_0}} \leq \epsilon$.

Now for any $m \geq 1$,

$$\begin{aligned}
a_{n_{k_0}+m} &\leq a_{n_{k_0}+m-1} - \alpha_{n_{k_0}+m-1}[\Phi(a_{n_{k_0}+m-1}) - \beta_{n_{k_0}+m-1}] \\
&\leq a_{n_{k_0}+m-2} - \alpha_{n_{k_0}+m-2}[\Phi(a_{n_{k_0}+m-2}) - b_{n_{k_0}+m-2}] \\
&\quad \alpha_{n_{k_0}+m-1}[\Phi(a_{n_{k_0}+m-1}) - \beta_{n_{k_0}+m-1}] \\
&. \\
&. \\
&. \\
&\leq a_{n_{k_0}} - \sum_{j=n_{k_0}}^{n_{k_0}+m-1} \alpha_{n_{k_0}}[\Phi(a_{n_{k_0}}) - \beta_{n_{k_0}}] \\
&\leq a_{n_{k_0}} \\
&< \epsilon.
\end{aligned}$$

Lemma 2.1.10: Let $\delta_j \in [0,1)\ \forall\ j \geq 1$. Then $\prod_{j=1}^{\infty}(1+\delta_j) < \infty \iff \sum_{j=1}^{\infty}\delta_j < \infty$

Proof: $(\Rightarrow)$ Let $\sum_{j=1}^{\infty}\delta_j < \infty$. From

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...,$$

then for any δ_j, $j \geq 1$

$$(1+\delta_j) \leq e^{\delta_j}.$$

Hence,

$$\prod_{j=1}^{\infty}(1+\delta_j) \le \prod_{j=1}^{\infty} e^{\delta_j} = e^{\delta_1}e^{\delta_2}e^{\delta_3} + ... = e^{\sum\limits_{j=1}^{\infty}\delta_j} \le e^m < \infty,$$

where $m = \sum\limits_{j=1}^{\infty}\delta_j$

($\Leftarrow$) Let $\prod\limits_{j=1}^{\infty}(1+\delta_j) < \infty$. Then

$$\sum_{j=1}^{\infty}\delta_j \le \prod_{j=1}^{\infty}(1+\delta_j) < \infty.$$

Lemma 2.1.11: Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty} \ge 0 \forall n$. Let

$$a_{n+1} \le (1+\alpha_n)a_n + b_n, n \ge 1.$$

If $\sum\limits_{n=1}^{\infty}\alpha_n < \infty$, $\sum\limits_{n=1}^{\infty} b_n < \infty$, then $\lim\limits_{n\to\infty} a_n$ exists.

Proof:

$$\begin{aligned}
a_{n+1} &\le (1+\alpha_n)a_n + b_n \\
&\le (1+\alpha_n)[(1+\alpha_{n-1})a_{n-1} + b_{n-1}] + b_n \\
&\le (1+\alpha_n)(1+\alpha_{n-1})(1+\alpha_{n-2})a_{n-2} \\
&\quad +(1+\alpha_n))(1+\alpha_{n-1})b_{n-2} \\
&\quad +(1+\alpha_n)b_{n-1} + b_n \\
&. \\
&. \\
&. \\
&\le \prod_{j=1}^{n}(1+\alpha_j)a_1 + \sum_{j+1}^{n} b_j \prod_{j=1}^{n}(1+\alpha_{j+1}) \\
&\le M,
\end{aligned}$$

so that a_n is bounded. We then write

$$a_{n+1} \le a_n + \gamma_n$$

where $\gamma_n = m\alpha_n + b_n$. Clearly $\sum\limits_{n+1}^{\infty}\gamma_n < \infty$ so that from Lemma 2.1.6 $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 2.1.12 ([30]) :Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n + \gamma_n, \quad n \geq 0 \tag{2.5}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$, and $\{\gamma_n\}_{n=1}^{\infty}$ satisfy the conditions:

(i) $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$, $\sum \alpha_n = \infty$ or equivalently $\prod_{n=1}^{\infty}(1-\alpha_n) = 0$;

(ii) $\limsup\limits_{n\to\infty}\beta_n \leq 0$;

(iii) $\gamma_n \geq 0, (n \geq 0)$, $\sum\limits_{n=1}^{\infty}\gamma_n < \infty$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

Proof. For any $\epsilon > 0$, let N be an integer big enough so that
$\beta_n < \frac{\epsilon}{2}$, $\sum\limits_{n=N}^{\infty}\gamma_n < \frac{\epsilon}{2}, n \geq N$.
Using equation (2.5) above and mathematical induction, we obtain for $n > N$

$$\begin{aligned}
a_{n+1} &\leq (1-\alpha_n)a_n + \alpha_n\beta_n + \gamma_n \\
&\leq (1-\alpha_n)[(1-\alpha_{n-1})a_{n-1} + \alpha_{n-1}\beta_{n-1} + \gamma_{n-1}] + \alpha_n\beta_n + \gamma_n \\
&= (1-\alpha_{n-1})(1-\alpha_n)a_{n-1} + \alpha_{n-1}(1-\alpha_n)\beta_{n-1} + (1-\alpha_n)\gamma_{n-1} \\
&\quad + \alpha_n\beta_n + \gamma_n \\
&\leq (1-\alpha_{n-1})(1-\alpha_n)a_{n-1} + \alpha_{n-1}(1-\alpha_n)\beta_{n-1} + \gamma_{n-1} + \gamma_n \\
&\leq (1-\alpha_{n-1})(1-\alpha_n)[(1-\alpha_{n-2})a_{n-2} + \alpha_{n-2}\beta_{n-2} + \gamma_{n-2}] \\
&\quad + \alpha_{n-1}(1-\alpha_n)\beta_{n-1} + \gamma_{n-1} + \gamma_n \\
&= (1-\alpha_{n-1})(1-\alpha_n)(1-\alpha_{n-2})a_{n-2} + (1-\alpha_{n-1})(1-\alpha_n)\alpha_{n-2}\beta_{n-2} \\
&\quad + (1-\alpha_{n-1})(1-\alpha_n)\gamma_{n-2} + \alpha_{n-1}(1-\alpha_n)\beta_{n-1} + \gamma_{n-1} + \gamma_n \\
&\leq (1-\alpha_{n-2})(1-\alpha_{n-1})(1-\alpha_n)a_{n-2} + \alpha_{n-2}(1-\alpha_{n-1})(1-\alpha_n)\beta_{n-2} \\
&\quad + \gamma_{n-2} + \gamma_{n-1} + \gamma_n \\
&\vdots \\
&\leq 1-\alpha_N(1-\alpha_{N+1})...(1-\alpha_n)a_N + \alpha_N(1-\alpha_{N+1})(1-\alpha_{N+2})...(1-\alpha_n)\beta_N \\
&\quad + \gamma_N + \gamma_{N+1}... + \gamma_n
\end{aligned}$$

$$\begin{aligned} &= \Big(\prod_{k=N}^{n}(1-\alpha_k)\Big)a_N + (1-(1-\alpha_N)(1-\alpha_{N+1})(1-\alpha_{N+2})...(1-\alpha_n))\beta_N \\ &\quad + \sum_{k=N}^{n}\gamma_k \\ &\le \prod_{k=N}^{n}(1-\alpha_k)a_N + \Big(1-\prod_{k=N}^{n}(1-\alpha_k)\Big)\beta_N + \sum_{k=N}^{n}\gamma_k. \end{aligned}$$

Therefore,

$$\begin{aligned} a_{n+1} &\le \prod_{k=N}^{n}(1-\alpha_k)a_N + \Big(1-\prod_{k=N}^{n}(1-\alpha_k)\Big)\frac{\epsilon}{2} \\ &\quad + \sum_{k=N}^{n}\gamma_k. \end{aligned}$$

Thus $\limsup_{n\to\infty} a_{n+1} \le 0 + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ is arbitrary, we have $\limsup_{n\to\infty} a_n = 0$. Thus $\lim_{n\to\infty} a_n = 0$.

2.2 Some Definitions and Results Used in the Main Result

In this section we state some known results and vital definitions which will be needed in establishing the main result of this work.

Definition.2.2.1. Let E be a normed space; $\emptyset \neq A \subset E$. A map $T : A \to A$ is said to be uniformly L-Lipchitzian if,

$$\|T^n x - T^n y\| \le L\|x-y\|, \quad \forall x, y \in A, \quad \forall n \ge 1, \quad L > 0.$$

Obviously, if T is asymptotically nonexpansive then T is uniformly L- Lipchitzian with $L = \sup_n\{k_n\}$

Definition.2.2.2. Let E be a Banach space. A map T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at a point $x \in D(T)$ if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence $\in E$ which converges weakly to a point $y \in E$ and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to x, then $Ty = x$.

Lemma 2.2.3 (Demi-closed principle in real Hilbert space). Let H be a real Hilbert space and K a nonempty closed and convex subset of H. Let $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$. then $I - T$ is demi-closed at zero,i.e.,for each sequence $\{x_n\}$ in K, if the sequence $\{x_n\}$ converges weakly to $x \in K$ and $\{(I-T)x_n\}$ converges strongly to 0, then $(I-T)x = 0$.

Proof Let $L = \sup_n k_n$

$$\begin{aligned} \|x_n - T^m x_n\| &\leq \|x_n - Tx_n\| + \|x_n - T^2 x_n\| \\ &\quad +... + \|T^{m-1}x_n - T^m x_n\| \\ &\leq mL\|x_n - Tx_n\| \to \infty. \end{aligned}$$

Define

$$f : H \to [0, \infty).$$

by,

$$f(y) := \limsup_{n\to\infty} \|x_n - y\|^2.$$

$$\begin{aligned} f(y) &= \limsup_{n\to\infty} \|x_n - y\|^2 \\ &= \limsup_{n\to\infty} \|x_n + x - x - y\|^2 \\ &= \limsup_{n\to\infty} [\|x_n - x\|^2 + 2\langle x_n - x, x - y\rangle + \|x - y\|^2] \\ &= \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2 \\ &= f(x) + \|x - y\|^2. \end{aligned} \tag{2.2.1}$$

$$\begin{aligned} f(T^m x) &= \limsup_{n\to\infty} \|x_n - T^m x\|^2 \\ &= \limsup_{n\to\infty} \|x_n - T^m x_n + T^m x_n - T^m x\|^2 \\ &\leq \limsup_{n\to\infty} [\|x_n - T^m x_n\|^2 + 2\langle x_n - T^m x_n, T^m x_n - T^m x\rangle + k_n^2\|x_n - x\|^2] \\ &\leq \limsup_{n\to\infty} [\|x_n - T^m x_n\|^2 + 2k_m\|x_n - T^m x_n\|\|x_n - x\| + k_m^2\|x_n - x\|^2] \\ &= \limsup_{n\to\infty} k_m^2\|x_n - x\|^2] \\ &= k_m^2 f(x). \end{aligned} \tag{2.2.2}$$

From (2.2.1) and (2.2.2) we have,

$$f(T^m x) = f(x) + \|T^m x - x\|^2 \leq k_m^2 f(x),$$

which implies,

$$\|T^m x - x\|^2 \leq (k_m^2 - 1) f(x) \to 0.$$

$$\Rightarrow T^m x \to x,$$

$$\Rightarrow T^{m+1} x \to x,$$

and by the continuity of T

$$T^{m+1} x \to Tx.$$

And by the uniqueness of limit we have,

$$Tx = x.$$

Lemma 2.2.4([21]lemma 1.3). Let E be a Hilbert space; $\alpha \in [0,1]; z, w \in E$. Then,

$$\|\alpha z + (1-\alpha)w\|^2 + \alpha(1-\alpha)\|z - w\|^2 = \alpha\|z\|^2 + (1-\alpha)\|w\|^2.$$

Proof.

$$\begin{aligned}
\|\alpha z + (1-\alpha)w\|^2 &= \alpha^2\|z\|^2 + (1-\alpha)^2\|w\|^2 + 2\alpha(1-\alpha)Re\langle z, w\rangle \\
&= \alpha^2\|z\|^2 + (1-\alpha)^2\|w\|^2 + \alpha(1-\alpha)[\|z\|^2 + \|w\|^2 - \|z-w\|^2] \\
&= \alpha^2\|z\|^2 + (1-\alpha)^2\|w\|^2 + \alpha(1-\alpha)\|z\|^2 + \alpha(1-\alpha)\|w\|^2 \\
&\quad -\alpha(1-\alpha)\|z-w\|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\alpha z + (1-\alpha)w\|^2 + \alpha(1-\alpha)\|z-w\|^2 &= \alpha^2\|z\|^2 + \alpha(1-\alpha)\|z\|^2 \\
&\quad +(1-\alpha)^2\|w\|^2 + \alpha(1-\alpha)\|w\|^2 \\
&= \alpha\|z\|^2 + (1-\alpha)\|w\|^2.
\end{aligned}$$

Lemma 2.2.5. Let H be a Hilbert space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in H, then $\{x_n\} \rightharpoonup x$ if and only if $\langle x_n, z\rangle \to \langle x, z\rangle, \quad \forall z \in H$.

Proof ($\Longrightarrow$) Suppose $\rightharpoonup x$, then $f(x_n) \to f(x), \quad \forall f \in H^*$ where H^* is the dual of H. Now let $z \in H$ be arbitrary (but fixed), then define $g : H \to \mathbb{R}$ or $\mathbb{C}$ by $g(x) = \langle x, z\rangle$, then $g \in H^*$. let x,y $\in H$ and $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$,

$$\begin{aligned} g(\alpha x + \beta y) &= \langle \alpha x + \beta y, z\rangle \\ &= \alpha\langle x, z\rangle + \beta\langle y, z\rangle \\ &= \alpha g(x) + \beta g(y). \end{aligned}$$

and

$$|g(x)| = |\langle x, z\rangle| \leq \|x\|\|z\| = \|z\|\|x\|.$$

Therefore $g \in H^*$ and since $x_n \rightharpoonup x$ then, $g(x_n) \to g(x)$ which implies $\langle x_n, z\rangle \to \langle x, z\rangle$ or $\langle x_n - x, z\rangle \to 0$.

($\Leftarrow$) Suppose $\langle x_n, z\rangle \to \langle x, z\rangle, \quad \forall z \in H$, we show that $x_n \rightharpoonup x$. Let $f \in H^*$ be arbitrary, then by the Riesz representation theorem there exists a unique $z \in H$ such that $f(x) = \langle x, z\rangle, \quad \forall x \in H$.

Thus, $f(x_n) = \langle x_n, z\rangle$ and since $\langle x_n, z\rangle \to \langle x, z\rangle$, we have $f(x_n) \to f(x)$ and since $f \in H^*$ is arbitrary,it follows that,

$$x_n \rightharpoonup x.$$

Lemma 2.2.6 Let H be a real Hilbert space,then for all $x, y \in H$,

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \qquad \text{(i)},$$

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2. \qquad \text{(ii)}$$

Proof. (i)

$$\begin{aligned}\|x-y\|^2 &= \langle x-y, x-y\rangle \\ &= \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \\ &= \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle \\ &= \|x\|^2 + \|y\|^2 - 2\langle x,y\rangle.\end{aligned}$$

Similarly for (ii).

2.3 Projection on convex subsets of a Hilbert Space

We discuss here the concept of projection operator on convex sets and some of its properties which are of vital importance in our work.

Definition.2.3.1: Let H be a Hilbert space and K a nonempty closed convex subset of H. For $x \in H$, by projection of x on K, we mean the element $z \in K$ denoted by $P_K(x)$ such that

$$\|x - P_K(x)\| \leq \|x-y\| \ \forall y \in K$$

or

$$\|x-z\| \leq \inf_{y\in K} \|x-y\|.$$

An operator on H into K, denoted by P_K, is called the *projection operator* if $P_K x = z$,where z is the projection of x on K.

Proposition 2.3.2. Let M be a closed convex subset of a Hilbert space H and $\rho = \inf_{y\in K}\|y\|$. Then there exists a unique $x \in M$ such that $\|x\| = \rho$.

Proof. Since M is a convex subset of H,then for $\alpha \in [0,1]$, $\alpha x + (1-\alpha)y \in M \ \forall x,y \in M$. By the definition of ρ, there exists a sequence of vectors $\{x_n\}$ in M such that $\|x_n\| \to \rho$. For $x = x_n$, $y = x_m$ and $\alpha = \frac{1}{2}$, we have,

$$\frac{x_n + x_m}{2} \in M$$

and

$$\|\frac{x_n + x_m}{2}\| \geq \rho.$$

Hence

$$\|x_n + x_m\| \geq \rho.$$

By the parallelogram law for elements x_n and x_m,

$$\|x_n + x_m\|^2 + \|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2.$$

Or

$$\begin{aligned} \|x_n - x_m\|^2 &= 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2 \\ &< 2\|x_n\|^2 + 2\|x_m\|^2 - 4\rho^2. \end{aligned}$$

Since,

$$2\|x_n\|^2 \to 2\rho^2$$

and

$$2\|x_m\|^2 \to 2\rho^2,$$

we have

$$2\|x_n - x_m\|^2 \to 2\rho^2 + 2\rho^2 - 4\rho_2 = 0, n, m \to \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence sequence in M. Since M is a closed subset of a Banach space H,it is complete(every closed subset of a complete metric space is complete). Thus $\{x_n\}$ is convergent in M i.e. there exists a vector $x \in M$ such that $\lim_{n\to\infty} x_n = x$. Since the norm is a continuous function, we Have,

$$\rho = \lim_{n=\infty} \|x_n\| = \|\lim_{n=\infty} x_n\| = \|x\|.$$

Thus, x is an element with the desired property.Now we show that x is unique. Let x_1 be another element of M such that $\rho = \|x_1\|$. Since $x, x_1 \in M$ and M is convex, $\frac{x + x_1}{2} \in M$. By the parallelogram law for the elements $\frac{x}{2}$ and $\frac{x_1}{2}$, we have,

$$\|\frac{x}{2} + \frac{x_1}{2}\|^2 + \|\frac{x}{2} - \frac{x_1}{2}\|^2 = \frac{2}{4}\|x\|^2 + \frac{2}{4}\|x_1\|^2.$$

Or

$$\begin{aligned} \|\frac{x+x_1}{2}\|^2 &= \frac{\|x\|^2}{2}+\frac{\|x_1\|^2}{2}-\frac{\|x-x_1\|^2}{2} \\ &< \frac{\|x\|^2}{2}+\frac{\|x_1\|^2}{2}=\rho^2. \end{aligned}$$

Or

$$\|\frac{x+x_1}{2}\|^2 < \rho.$$

Which contradicts the definition of ρ hence x is unique.

Proposition 2.3.3. Let M be a closed subspace of a Hilbert space H. Let $x \notin M$ and ρ be the distance between x and M i.e. $\rho = \inf_{u\in M} \|x-u\|$. Then there exists a unique vector $w \in M$ such that $\|x-w\| = \rho$

Proof: The set $N = x + M = \{x + v : v \in M\}$ is a closed convex subset of H, and $\rho = \inf_{x+v\in M} \|0-(x+v)\|$ is the distance of 0 from N. Since $-v \in M$ for all $v \in M$, $\rho = \inf_{v\in M} \|0-(x+v)\|$. By proposition(1), there exists a unique vector $u \in M$ such that $\rho = \|u\|$. The vector $w = x-u = -(u-x)$ belongs to M (As we have $M = N-x = \{z-x : z \in N, x \notin M\}$ and M is a subspace therefore $-(u-x) \in M$ thus $\|x-w\| = \|u\| = \rho$. w is unique, for if w is not unique, $w_1 \neq w$ is a vector in M such that $\|x-w_1\| = \rho$. This implies that $u_1 = (x-w_1)$ is a vector in N such that $\|u_1\| = \|x-w_1\| = \rho$.This contradicts that u_1 is unique. Hence, w is unique.

Remark 2.3.1: proposition(2.3.3) can be rephrased as follows: Let M be a closed convex subset of a Hilbert space H and for a any $x \in H$, let $\rho = \inf_{u\in M} \|x-u\|$ then there exists a unique element $w \in M$ such that $\rho = \|x-w\|$.

Proposition 2.3.4. Let K be a closed convex subset of a Hilbert space H. Then for any $x \in H$, there is a unique element in K closest to x; that is a unique element $P_K x = z \in K$ such that,

$$\|x-z\| \leq \|x-y\|, \forall y \in K$$

Or

$$\|x-z\| \leq \inf_{y\in K} \|x-y\|.$$

Proof: Since the set $x - K$ consisting of all elements of the form $x - y$ for all $y \in K$ is a closed convex set, it follows from proposition(2) and remark(1)that there is a unique $z \in K$ such that

$$\|x - z\| \leq \|x - y\|, \forall y \in K.$$

Proposition 2.3.5 (Variational characterization of projections). Let K be a closed convex set in a Hilbert space H and let $P_K : H \to K$ be the projection of H onto K. Then for any $x \in H, P_K x = z \in K$ if and only if $Re\langle x - z, y - z\rangle \leq 0, \forall y \in k$.

Proof: Let z be the projection of x into K. Then for any $\alpha, 0 < \alpha \leq 1$, since K is convex, $v = \alpha y + (1 - \alpha)z \in K$ for all $y \in K$.Therefore,

$$\|x - v\| \geq \|x - z\|,$$

and since,

$$\begin{aligned} \|x - v\|^2 &= \|x - (\alpha y + (1 - \alpha)z)\|^2 \\ &= \|x - z\|^2 + \alpha^2\|y - z\|^2 - 2Re\langle x - z, y - z\rangle. \end{aligned}$$

$$\begin{aligned} Re\langle x - z, y - z\rangle &= \frac{1}{2\alpha}(\|x - z\|^2 - \|x - v\|^2) + \frac{\alpha}{2}\|y - z\|^2 \\ &\leq \frac{\alpha}{2}\|y - z\|^2. \end{aligned}$$

Taking limit as $\alpha \to 0$ we have,

$$Re\langle x - z, y - z\rangle \leq 0.$$

Conversely, assume that $z \in K$ satisfies the condition $Re\langle x - z, y - z\rangle \leq 0 \quad \forall y \in K.$, then

$$\|x - y\|^2 = \|x - z\|^2 + \|y - z\|^2 - 2Re\langle x - z, y - z\rangle \geq \|x - z\|^2.$$

i.e.

$$\|x - z\| = \inf \|x - y\| \quad \forall y \in K.$$

Which means that $z = P_K x$

Proposition 2.3.6 The projection map P_K defined on a Hilbert space H onto its non empty closed convex subset K is non-expansive i.e.

$$\|P_K(u) - P_K(v)\| \leq \|u - v\|, \qquad \forall u, v \in K.$$

Proof: In view of proposition(2.3.4)

$$Re\langle P_K(u) - u, P_K(u) - v\rangle \leq 0, \qquad \forall v \in K \tag{2.4.1}$$

Put $u = u_1$ in (2.4.1) to get

$$Re\langle P_K(u_1) - u_1, P_K(u_1) - v\rangle \leq 0, \qquad \forall v \in K \tag{2.4.2}$$

Put $u = u_2$ (2.4,1) to get

$$Re\langle P_K(u_2) - u_2, P_K(u_2) - v\rangle \leq 0, \qquad \forall v \in K \tag{2.4.3}$$

Since $P_K(u_2)$ and $P_K(u_1) \in K$, choosing $v = P_K(u_2)$ and $v = P_K(u_1)$ respectively in (2.4.3) and (2.4.2) we get

$$Re\langle P_K(u_1) - u_1, P_K(u_1) - P_K(u_2)\rangle \leq 0, \tag{2.4.4}$$

$$Re\langle P_K(u_2) - u_2, P_K(u_2) - P_K(u_1)\rangle \leq 0, \tag{2.4.5}$$

From (2.4.4) and (2.4.5) we get,

$$Re\langle P_K(u_1) - u_1 - P_K(u_2) + u_2, P_K(u_1) - P_K(u_2)\rangle \leq 0.$$

Or

$$Re\langle P_K(u_1) - P_K(u_2), P_K(u_1) - P_K(u_2)\rangle \leq Re\langle u_1 - u_2, P_K(u_1) - P_K(u_2)\rangle.$$

Or

$$\|P_K(u_1) - P_K(u_2)\|^2 \leq Re\langle u_1 - u_2, P_K(u_1) - P_K(u_2)\rangle.$$

and by the Cauchy Schwartz Bunyakowski inequality we

$$\|P_K(u_1) - P_K(u_2)\|^2 \leq \|u_1 - u_2\|\|P_K(u_1) - P_K(u_2)\|.$$

Or

$$\|P_K(u_1) - P_K(u_2)\| \leq \|u_1 - u_2\|.$$

Chapter 3

Strong Convergence of Modified Averaging Iterative Algorithm for Asymptotically Nonexpansive Maps

3.1 Main Result

In this section we introduce a modified Mann iteration scheme for asymptotically nonexpansive map and prove the strong convergence to a fixed point of asymptotically nonexpansive map in real Hilbert spaces.

Algorithm: Let H be a real Hilbert space and let K be a nonempty, closed and convex subset of H. Let $T : K \to K$ be a non linear map. Let $\{\alpha_n\}$ and $\{t_n\}$ be two sequences in[0,1]. For any given $x_0 \in K$,let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by,

$$\begin{cases} y_n = P_K[(1-t_n)x_n] \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T^n y_n, n \geq 0. \end{cases} \tag{3.1.1}$$

Theorem 3.1.1: Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let $T : K \to K$ be an asymptotically nonepansive map with $k_n \in [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$ and $Fix(T) \neq \emptyset$, and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers in [0,1]. Suppose the following conditions are satisfied,

(I)$\lim_{n\to\infty} t_n = 0$,

(II)$\sum_{n=0}^{\infty} t_n = \infty$,

(III)$\alpha_n \in (a,b) \subset (0,1)$,

(IV)$\sum_{n=0}^{\infty}(k_n - 1) < \infty$,

(V) $t_n \geq \dfrac{k_n^2 - 1}{k_n^2}$,

then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.1.1) converges strongly to a fixed point of T.

A prototype example of $\{t_n\}$ is the sequence $\{\dfrac{k_n^2-1}{k_n^2} + \dfrac{1}{k^2 n}\}$, where $k = \sup_n k_n$ and a prototype of $\{\alpha_n\}$ is the sequence $\{0.9 - \dfrac{2}{4n-1}\}$.

Proof of Theorem

First we establish the boundedness of $\{x_n\}$ and $\{y_n\}$. Let $p \in Fix(T)$ be arbitrary,

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)(y_n - p) + \alpha_n(T^n y_n - p)\| \\ &\leq (1-\alpha_n)\|y_n - p\| + \alpha_n k_n \|y_n - p\| \\ &= \|y_n - p\| + \alpha_n(k_n - 1)\|y_n - p\| \\ &\leq [1 + (k_n - 1)]\|y_n - p\| \\ &\leq e^{(k_n - 1)}\|y_n - p\|. \end{aligned}$$

But

$$\begin{aligned} \|y_n - p\| &= \|P_K[(1-t_n)x_n] - p\| \\ &\leq \|(1-t_n)x_n - p\| \\ &\leq (1-t_n)\|x_n - p\| + t_n\|p\| \\ &\leq \max\{\|x_n - p\|, \|p\|\}. \end{aligned}$$

Therefore,

$$\|x_{n+1} - p\| \leq e^{(k_n - 1)} \max\{\|x_n - p\|, \|p\|\} \forall n \geq 0. \tag{3.1.2}$$

Claim,

$$\|x_{n+1} - p\| \leq e^{\sum_{j=0}^{n}(k_j - 1)} \max\{\|x_n - p\|, \|p\|\} \forall n \geq 0.$$

We now establish our claim by induction, for n=0, from (3.1.2) we have,

$$\|x_1 - p\| \leq e^{(k_0-1)} \max\{\|x_0 - p\|, \|p\|\}.$$

then assume true for n=i, that is,

$$\|x_{i+1} - p\| \leq e^{\sum_{j=0}^{i}(k_j-1)} \max\{\|x_0 - p\|, \|p\|\}.$$

Then for n=i+1,

$$\begin{aligned}
\|x_{i+2} - p\| &\leq e^{(k_{i+1}-1)} \max\{\|x_{i+1} - p\|, \|p\|\} \\
&\leq e^{(k_{i+1}-1)} \max\{e^{\sum_{j=0}^{i}(k_j-1)} \max\{\|x_0 - p\|, \|p\|\}, \|p\|\} \\
&\leq e^{\sum_{j=0}^{i+1}(k_j-1)} \max\{\max\{\|x_0 - p\|, \|p\|\}, \|p\|\} \\
&= e^{\sum_{j=0}^{i+1}(k_j-1)} \max\{\|x_0 - p\|, \|p\|\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x_{n+1} - p\| &\leq e^{\sum_{j=0}^{n+1}(k_j-1)} \max\{\|x_0 - p\|, \|p\|\} \\
&\leq e^{\sum_{j=0}^{\infty}(k_j-1)} \max\{\|x_0 - p\|, \|p\|\}.
\end{aligned}$$

But,

$$\|x_0 - p\| \leq e^{\sum_{j=0}^{n}(k_j-1)} \max\{\|x_0 - p\|, \|p\|\}.$$

Therefore,

$$\|x_n - p\| \leq e^{\sum_{j=0}^{n}(k_j-1)} \max\{\|x_0 - p\|, \|p\|\} \forall n \geq 0.$$

Hence $\{x_n\}$ is bounded and,

$$\begin{aligned}
\|y_n - p\| &\leq \max\{\|x_n - p\|, \|p\|\} \\
&\leq \max\{e^{\sum_{j=0}^{\infty}(k_j-1)} \max\{\|x_0 - p\|, \|p\|\}, \|p\|\} \\
&= e^{\sum_{j=0}^{\infty}(k_j-1)} \max\{\|x_0 - p\|, \|p\|\}.
\end{aligned}$$

so, $\{y_n\}$ is also bounded. Now,

$$\|T^n x_n - p\| \leq k_n \|x_n - p\|.$$

And,

$$\|T^n y_n - p\| \leq k_n \|y_n - p\|.$$

Therefore $\{T^n x_n\}$ and $\{T^n y_n\}$ are bounded.

From (3.1.1) we have,

$$y_n - T^n y_n = \frac{1}{\alpha_n}(y_n - x_{n+1}). \tag{3.1.3}$$

In Lemma 2.2.4, if we let, $z = T^n y_n - p$ and $w = y_n - p$, we deduce that

$$\begin{aligned}
\|x_{n+1} - p\|^2 + \alpha_n(1-\alpha_n)\|T^n y_n - y_n\|^2 &= \alpha_n \|T^n y_n - T^n p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\leq \alpha_n k_n^2 \|y_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&= \|y_n - p\|^2 + \alpha_n (k_n^2 - 1)\|y_n - p\|^2 \\
&= \|P_K[(1-t_n)x_n] - p\|^2 \\
&\quad + \alpha_n (k_n^2 - 1)\|P_K[(1-t_n)x_n] - p\|^2 \\
&\leq \|(1-t_n)x_n - p\|^2 + \alpha_n (k_n^2 - 1)\|(1-t_n)x_n - p\|^2 \\
&= \|x_n - p - t_n x_n\|^2 + \alpha_n (k_n^2 - 1)\|x_n - p - t_n x_n\|^2 \\
&= \|x_n - p\|^2 - 2t_n \langle x_n - p, x_n \rangle \\
&\quad + t_n^2 \|x_n\|^2 + \alpha_n (k_n^2 - 1)\|x_n - p\|^2 \\
&\quad - 2t_n \alpha_n (k_n^2 - 1)\langle x_n - p, x_n \rangle + t_n^2 \alpha_n (k_n^2 - 1)\|x_n\|^2 \\
&\leq k_n^2 \|x_n - p\|^2 + t_n[-2\langle x_n - p, x_n \rangle \\
&\quad - 2\alpha_n (k_n^2 - 1)\langle x_n - p, x_n \rangle + t_n k_n^2 \|x_n\|^2].
\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists $M > 0$ such that

$$-2\langle x_n - p, x_n \rangle - 2\alpha_n (k_n^2 - 1)\langle x_n - p, x_n \rangle + t_n k_n^2 \|x_n\|^2 \leq M.$$

Therefore,

$$\|x_{n+1}-p\|^2 - k_n^2\|x_n-p\|^2 + (1-\alpha_n)\|y_n - x_{n+1}\|^2 \le t_n M. \tag{3.1.4}$$

Now we consider two cases, to prove that $\{x_n\}$ converges strongly to a fixed point p of T.

Case 1. Suppose that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing sequence. Then $\{\|x_n - p\|\}$ is convergent. Clearly,

$$\|x_{n+1}-p\|^2 - k_n^2\|x_n-p\|^2 \to 0.$$

this together with condition (I) and (3.1.4) imply that

$$\|y_n - x_{n+1}\| \to 0. \tag{3.1.5}$$

Also, we note that

$$\begin{aligned}\|y_n - x_n\| &= \|P_K[(1-t_n)x_n] - x_n\| \\ &\le \|(1-t_n)x_n - x_n\| \\ &= t_n\|x_n\| \to 0. \end{aligned} \tag{3.1.6}$$

Hence,

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|x_{n+1} - y_n + y_n - x_n\| \\ &\le \|x_{n+1} - y_n\| + \|y_n - x_n\| \to 0, n \to \infty. \end{aligned} \tag{3.1.7}$$

and

$$\|y_n - T^n y_n\| = \frac{1}{\alpha_n}\|y_n - x_{n+1}\| \to 0, n \to \infty. \tag{3.1.8}$$

From (3.1.6) and (3.1.8), we have

$$\begin{aligned}\|x_n - T^n x_n\| &\le \|x_n - y_n\| + \|y_n - T^n y_n\| + \|T^n y_n - T^n x_n\| \\ &\le (1+k_n)\|x_n - y_n\| + \|y_n - T^n y_n\| \to 0, n \to \infty. \end{aligned} \tag{3.1.9}$$

Now,

$$\begin{aligned}
\|Tx_n - x_n\| &= \|Tx_n - T^n x_n + T^n x_n - x_n\| \\
&\leq \|x_n - T^n x_n\| + L\|T^{n-1}x_n - x_n\| \\
&\leq \|x_n - T^n x_n\| + L^2\|x_n - x_{n-1}\| + L\|T^{n-1}x_{n-1} - x_n\| \\
&\leq \|x_n - T^n x_n\| + L^2\|x_n - x_{n-1}\| + L\|T^{n-1}x_{n-1} - x_{n-1}\| \\
&\quad +L\|x_{n-1} - x_n\| \to 0, n \to \infty. \qquad (3.1.10)
\end{aligned}$$

Since $I - T$ is demiclosed, we now prove that $\{x_n\}$ and $\{y_n\}$ converge weakly to a fixed point p of T. First we prove that $w_w(x_n)$ is a nonempty set and,

$$w_w(x_n) \subseteq Fix(T). \qquad (3.1.11)$$

where $w_w(x_n)$ is the weak w-limit set of $\{x_n\}$ defined by $w_w(x_n) := \{y \in H : y = w - \lim_{k\to\infty} x_{n_k}$ for some $n_k \to \infty$.

Now since $\{x_n\}, \{y_n\}, \{T^n x_n\}$, and $\{T^n y_n\}$ are all bounded sequences,there exists $r > 0$ such that $\{x_n\} \cup \{y_n\} \cup \{T^n x_n\} \cup \{T^n y_n\} \subseteq B(p, r) \cap K := C$,where $B(p, r)$ is a closed ball of H with center p and radius r and so C is a nonempty bounded closed convex subset of K. Since H is a Hilbert space,it is uniformly convex and C is a nonempty bounded closed convex subset of K and so C is weakly compact and weakly closed subset of K. This implies that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to a point $p \in w_w(x_n)$, which shows that $w_w(x_n)$ is nonempty. For any $p \in w_w(x_n)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \rightharpoonup p$. Again by (3.1.10), we have $\lim_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0$. Therefore, it follows from lemma(2.2.3) that $(I - T)p = 0$, that is $p \in Fix(T)$. The conclusion (3.1.11) is proved. Taking any $p \in w_w(x_n) \subseteq Fix(T)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that as $n_i \to \infty$,

$$x_{n_i} \rightharpoonup p. \qquad (3.1.12)$$

Hence, from (3.1.9)and (3.1.12), it follows that as $n_i \to \infty$,

$$T^{n_i} x_{n_i} = (T^{n_i} x_{n_i} - x_{n_i}) + x_{n_i} \rightharpoonup p.$$

Again from (3.1.1),condition(I) and (3.1.12), it follows that as $n_i \to \infty$,

$$y_{n_i} = P_K[x_{n_i} - t_{n_i} x_{n_i}] \rightharpoonup p.$$

. On the other hand,from (3.1.8) and (3.1.12),we have as $n_i \to \infty$,

$$T^{n_i} y_{n_i} = (T^{n_i} y_{n_i} - x_{n_i}) + x_{n_i} \rightharpoonup p. \tag{3.1.13}$$

By the same way, from (3.1.1),(3.1,8) and (3.1.12),

$$x_{n_i+1} = y_{n_i} - \alpha_{n_i}(y_{n_i} - T^{n_i} y_{n_i}) \rightharpoonup p. \tag{3.1.14}$$

Therefore, it follows from (3.1.9) and (3.1.14) that as $n_i \to \infty$,

$$T^{n_i+1} x_{n_i+1} = (T^{n_i+1} x_{n_i+1} - x_{n_i+1}) + x_{n_i+1} \rightharpoonup p. \tag{3.1.15}$$

By (3.1.1),(3.1.14) and condition(I),

$$y_{n_i+1} = P_K[x_{n_i+1} - t_{n_i+1} x_{n_i+1}] \rightharpoonup p. \tag{3.1.16}$$

Therefore, from (3.1.8) and (3.1.16) it follows that as $n_i \to \infty$,

$$T^{n_i+1} y_{n_i+1} = (T^{n_i+1} y_{n_i+1} - y_{n_i+1}) + y_{n_i+1} \rightharpoonup p. \tag{3.1.17}$$

Continuing in this way, we obtain by induction that,for any $m \geq 0$,

$$x_{n_i+m} \rightharpoonup p, y_{n_i+m} \rightharpoonup p.(n_i \to \infty).$$

$$T^{n_i+m} x_{n_i+m} \rightharpoonup p, T^{n_i+m} y_{n_i+m} \rightharpoonup p.(n_i \to \infty).$$

Next we prove that $x_n \rightharpoonup p$ as $(n_i \to \infty)$.

Observe that,

$$\{x_n\}_{n=n_1}^{\infty} = \lim_{k \to \infty} \cup_{m=0}^{k} \{x_{n_i+m}\}_{i=1}^{\infty}. \tag{3.1.18}$$

Since the sequences $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ both converge weakly to p as $n_i \to \infty$, the sequence $\cup_{m=0}^{1} \{x_{n_i+m}\}$ converges weakly to p. By induction, we have that, for any positive integer k,

$$\cup_{m=0}^{k} \{x_{n_i+m}\}_{i=1}^{\infty} \rightharpoonup p.$$

Letting $k \to \infty$ it follows from (3.1.18) that the sequence $\{x_n\}$ converges weakly to p and consequently $\{y_n\}$ converges weakly to a fixed point p of T.

Next we prove that $\{x_n\}$ and $\{y_n\}$ converge strongly to p.

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)y_n+\alpha_n T^n y_n - p\|^2 \\
&= \|(1-\alpha_n)(y_n-p)+\alpha_n(T^n y_n-p)\|^2 \\
&\leq [\|(1-\alpha_n)(y_n-p)\|+\|\alpha_n(T^n y_n-p)\|]^2 \\
&\leq [(1-\alpha_n)\|y_n-p\|+\alpha_n k_n\|y_n-p\|]^2 \\
&\leq [(1-\alpha_n)k_n\|y_n-p\|+\alpha_n k_n\|(y_n-p)\|]^2 \\
&= k_n^2\|y_n-p\|]^2 \\
&= k_n^2\|P_K[(1-t_n)x_n]-p\|^2 \\
&\leq k_n^2\|(1-t_n)x_n-p\|^2 \\
&= k_n^2(1-t_n)^2\|x_n-p\|^2 \\
&\quad -2k_n^2\langle(1-t_n)(x_n-p),t_n p\rangle+k_n^2t_n^2\|p\|^2 \\
&\leq (1-t_n)\|x_n-p\|^2-2t_nk_n^2\langle x_n-p,p\rangle \\
&\quad +2t_n^2k_n^2\langle x_n-p,p\rangle+k_n^2t_n^2\|p\|^2 \\
&= (1-t_n)\|x_n-p\|^2+t_n[-2k_n^2\langle x_n-p,p\rangle \\
&\quad +2t_nk_n^2\langle x_n-p,p\rangle+k_n^2t_n\|p\|^2]. \qquad (3.1.19)
\end{aligned}$$

Since $\{x_n\}$ converges weakly to p, thus by Lemma 2.2.5, we have that $\langle x_n-p,p\rangle \to 0$ and also we have $t_n\|p\|^2 \to 0$.

Hence,

$$\delta_n = -2k_n^2\langle x_n-p,p\rangle+2t_nk_n^2\langle x_n-p,p\rangle+k_n^2t_n\|p\|^2 \to 0.$$

Therefore applying Lemma 2.1.12 to (3.1.19), we obtain that $x_n \to p$ and consequently $y_n \to p$.

Case 2.

Assume that $\|x_n-p\|$ is not monotonically decreasing sequence. Set $Z_n = \|x_n-p\|^2$ and let $u : \mathbb{N} \to \mathbb{N}$ be a map for all $n \geq n_0$ (for some n_0 large enough) given by,

$$u(n) = \max\{k \in \mathbb{N} : k \leq n, Z_k \leq Z_{k+1}\}.$$

Clearly u is non decreasing sequence such that $u(n) \to \infty$ as $n \to \infty$ and $Z_{u(n)} \leq Z_{u(n)+1}$ for all $n \geq n_0$.

From (3.1.4) and (3.1.6), we have that,

$$\|x_{u(n)+1} - y_{u(n)}\|^2 \leq \frac{t_{u(n)}M}{1 - \alpha_{u(n)}} \to 0.$$

and

$$\|y_{u(n)} - x_{u(n)}\| \to 0.$$

Thus,

$$\|x_{u(n)+1} - x_{u(n)}\| \leq \|x_{u(n)+1} - y_{u(n)}\| + \|y_{u(n)} - x_{u(n)}\| \to 0.$$

By the same argument as (3.1.5)-(3.1.10) in case 1, we conclude immediately that $x_{u(n)}$ and $y_{u(n)}$ converge weakly to p as $u(n) \to \infty$. At the same time, we note that for all $n \geq n_0$

$$\begin{aligned} 0 &\leq \|x_{u(n)+1} - p\|^2 - \|x_{u(n)} - p\|^2 \\ &\leq t_{u(n)}[2k^2_{u(n)}\langle p - x_{u(n)}, p\rangle + 2t_{u(n)}k^2_{u(n)}\langle x_{u(n)} - p, p\rangle \\ &\quad +k^2_{u(n)}t_{u(n)}\|p\|^2] - \|x_{u(n)} - p\|^2]. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x_{u(n)} - p\|^2 &\leq 2k^2_{u(n)}\langle p - x_{u(n)}, p\rangle + k^2_{u(n)}t_{u(n)}\|p\|^2 \\ &\quad +2t_{u(n)}k^2_{u(n)}\langle x_{u(n)} - p, p\rangle. \end{aligned}$$

We deduce that,

$$\|x_{u(n)} - p\| \to 0,$$

therefore,

$$\lim_{n\to\infty} Z_{u(n)} = \lim_{n\to\infty} Z_{u(n)+1} = 0.$$

Furthermore, for $n \geq n_0$,it is easily observed that $Z_n \leq Z_{u(n)+1}$ if $n \neq u(n)$ (that is $u(n) < n$), because $Z_j > Z_{j+1}$ for $u(n)+1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq Z_n \leq \max\{Z_{u(n)}, Z_{u(n)+1}\} = Z_{u(n)+1}.$$

Therefore, $\lim_{n\to\infty} Z_n = 0$,that is $\{x_n\}$ converges strongly to p and consequently $\{y_n\}$ also converges strongly to p. This concludes the proof.

3.2 Applications

Corollary 3.2.1: Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let $T : K \to K$ be a nonepansive map with $Fix(T) \neq \emptyset$, and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers in [0,1]. Suppose the following conditions are satisfied

(I)$\lim_{n\to\infty} t_n = 0$,

(II)$\sum_{n=0}^{\infty} t_n = \infty$,

(III)$\alpha_n \in (a, b) \subset (0, 1)$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated from an arbitrary $x_1 \in K$

$$\begin{cases} y_n = P_K[(1-t_n)x_n] \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, n \geq 0. \end{cases} \tag{3.2.1}$$

converges strongly to a fixed point of T.

Corollary 3.2.2: Let H be a real Hilbert space and K a nonempty, closed and convex subset of H with $0 \in K$. Let $T : K \to K$ be an asymptotically nonepansive map with $Fix(T) \neq \emptyset$, and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers in [0,1]. Suppose the following conditions are satisfied

(I)$\lim_{n\to\infty} t_n = 0$,

(II)$\sum_{n=0}^{\infty} t_n = \infty$,

(III)$\alpha_n \in (a,b) \subset (0,1)$,

(V) $t_n \geq \dfrac{k_n^2 - 1}{k_n^2}$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated from an arbitrary $x_1 \in K$

$$\begin{cases} y_n = (1-t_n)x_n \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, n \geq 0. \end{cases} \tag{3.2.2}$$

converges strongly to a fixed point of T.

Corollary 3.2.3: Let H be a real Hilbert space. Let $T : H \to H$ be an asymptotically nonepansive map with $Fix(T) \neq \emptyset$, and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers in [0,1]. Suppose the following conditions are satisfied

(I)$\lim_{n\to\infty} t_n = 0$,

(II)$\sum_{n=0}^{\infty} t_n = \infty$,

(III)$\alpha_n \in (a,b) \subset (0,1)$,

(V) $t_n \geq \dfrac{k_n^2 - 1}{k_n^2}$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated from an arbitrary $x_1 \in K$

$$\begin{cases} y_n = (1-t_n)x_n \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, n \geq 0. \end{cases} \tag{3.2.3}$$

converges strongly to a fixed point of T.

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