BOUNDARY VALUE PROBLEMS FOR QUASILINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

BY

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Certification

This is to certify that this project titled Boundary Value Problems For Quasilinear Second Order Differential Equations is presented by Audu Abdulmalik Onubedo, a postgraduate student in the department of mathematics, faculty of physical sciences, University of Nigeria Nsukka with registration number PG/M.Sc/10/52736 for the award of master of science (M.Sc) degree in Mathematics.

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Dedication

To almighty ALLAH, my beloved parents (Mall Audu Owuda and Mariyamoh Oyewu) and my elder brothers Mr Clement Omeiza Audu and Mall Ahmed Ozigi Audu all of blessed memory.

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Contents

Abstract

This project is concerned with the review of some boundary value problems for nonlinear ordinary differential equations using topological and variational methods. A more classical boundary value problems for ordinary differential equations (like the boundary value problems on a ball, initial value problems, problems on annular domains and positone problems) which represent the main interest of a wide number of researchers in the world is studied.

Chapter 1

Introduction

The theory of differential equations is a field of mathematics that is motivated greatly by challenges arising from different applications, and leading to the birth of other fields of mathematics. It is not our intention to show a panoramic view of this enormous field, we only intend to reveal its relation to the theory of dynamical systems.The mathematical results related to the investigation of differential equation can be grouped as follows:

(a) Analytic and numerical methods for differential equations.

(b) Prove the existence and uniqueness of solutions of differential equations.

(c) Characterise the properties of solution without deriving explicit formulas for them.

There is obviously a significant demand coming from application, for results in the first direction. It is worthy to note that in the last century the emphasis was on the numerical approximation of solutions. The question of existence and uniqueness of initial value problem for ordinary differential equations was answered completely in the first half of the twentieth century ([43]), hence motivating the development of fixed point theorems in normed spaces.

Today's research is in the direction of existence and uniqueness of solutions of boundary value problems for non-linear ordinary differential equations where the exact number of positive solutions are required is an actively studied field . The studies in the third direction goes back to the end of the nineteenth century, when a considerable demand to investigate non-linear differential equation appeared, and it turned out that these kind of equations cannot be solved analytically in most of the cases particularly as in the case of boundary value problems for quasilinear second order differential equations.

In this work, we provide a survey of several results concerning solutions of quasilinear differential equations where the independent variable vary over domains such as a ball, an annular domain determined by concentric spheres to determine if it has solutions which only depend upon the radial variable. This is well illustrated by the classical problems of finding the eigenvalues and the eigenfunctions of an operator subject to some boundary conditions on the domain of operation. Further problems are concerned with the existence of positive solutions of the equation

$$[\phi(u')]' + \frac{N-1}{r}\phi(u') + g(\lambda, u) = 0, r \in (a,b), u(a) = u(b) = 0$$

where $\lambda > 0$ and Ω is a bounded domain in $\mathbb{R}^n$. In the case that $\Omega = \{x \in \mathbb{R}^n : 0 < a < |x| < b\}$, then the solutions of the above equation are solutions of the boundary value problem

$$u'' + \frac{N-1}{r}u' + f(u) = 0, r \in (a,b), u(a) = u(b) = 0, (a,b) \subset \Omega.$$

This follows from the maximum principle for elliptic equations that solutions can only assume positive values in the interior of the domain. For $N = 1$, this problem is amenable to reduction of order methods, and are explicitly solved as demonstrated in section 3.3 illustrative example infra.

To establish that a given eigenvalue problem has positive solutions, we will start with the fixed boundary conditions and try to find the equation (by finding an appropriate coefficient λ) that has a nonzero solution satisfying the given boundary conditions. This kind of reversed boundary value problem is called an eigenvalue problem discussed in subsection (4.3.2). The specific value(s) of λ that would give a solution of the boundary value problem is called an eigenvalue of the boundary value problem. The nonzero solution that arises from each eigenvalue is called a corresponding eigenfunction of the boundary value problem. A typical example was given in ([45]) and discussed in chapter 3, section 3.3 of this work.

Chapter 2

Literature Review

The subject of Differential Equations is a well established part of mathematics and its systematic development goes back to the early days of the development of Calculus. Many recent advances in mathematics, paralleled by a renewed and flourishing interaction between mathematics, the sciences, and engineering, have again shown that many phenomena in the applied sciences, modelled by differential equations will yield some mathematical explanation of these phenomena (at least in some approximate sense). The intent of this set of notes is to present several of the important existence theorems for solutions of various types of problems associated with differential equations and provide qualitative and quantitative descriptions of solutions. At the same time, we develop methods of analysis which may be applied to carry out the above and which have applications in many other areas of mathematics, as well. As methods and theories are developed, we shall also pay particular attention to illustrate how these findings may be used. As differential equations are equations which involve functions and their derivatives as unknowns, we shall observe that differential equations are equations in spaces of functions and therefore shall develop existence theories for equations defined in various types of function spaces, which usually will be function spaces that are in some sense natural for the given problem. E.D Rogak provides that Boundary value problems are of fundamental importance in physics. However, solving such problems usually involves a combination of methods from ordinary differential equations, functional analysis, complex functions, and measure theory. Also

G. Tesch (1991) consider two specializations of the general problem for quasilinear hyperbolic equations in the time-dependent (noncylindrical) domain Q for each $T > 0$ where Q is a region in euclidean space $\mathbb{R}^{n+1}$ that can be mapped smoothly onto the exterior of an infinite rigth circular cylinder. One physical model of a problem of this kind is that of scattering of acoustical waves by a moving body in space that also changes its shape with time. The main results are two nonequivalent existence theorems for the problem which provides at least one weak solution to the problem. The main tool used in the proof is the Leray-Schauder fixed-point theorem. This existence theorem permits considerable nonlinearity in functions but the hypotheses are not strong enough to yield uniqueness.

A. g. Kartsatos (2005) in his study of monotonicity of solutions in bounded solutions for quasilinear system

$$x' = A(t)x + f(t, x) \qquad (A)$$

made an assumption that includes inner products involving $A(t)$ or $f(t, u)$. This development is preferred because it constitutes a good towards the corresponding theory in Banach and Hilbert spaces. This theory encompasses substantial part of the mordern theory of ODE and even PDE. The study introduces norm properties in $\mathbb{R}^n$, examine the stability properties of solutions of (A) via monotonicity condition and the result given thus:

Lemma

Let $A, B \in M_n$ be given with A having all of its eigenvalues with negative real part and B positive definite. There exists a positive definite $V \in M_n$ such that $A^T V + VA = -B$ guaranteed a solution.

On bounded solution to the quasilinear system

$$x' = A(t, x)x = f(t, x) \qquad (B).$$

The following theorem guaranteed that the Zero solution is asymptotically stable.

Theorem

Assume the following for the system (B)

(i) $A : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow M_n$ is continuous

(ii)$f : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous and such that $\|f(t,x)\| \leq \lambda|x|, (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ where λ is a positive constant;

(iii) there exists a positive constant $K \in (0, \frac{1}{\lambda})$ with the property: for every $f \in c_n(\mathbb{R}_+)$ there exist a fundamental matrix $X_f(t)$ of the system

$x' = A(t, f(t))x$ such that $\int |x_f(t)x_f^{-1}(s)|ds \leq k, t \in \mathbb{R}_+$.

Now the challenge of this project is to examine solution of boundary value problems for quasilinear second order differential equations restricted to a ball, initial value problems, problems on annular domains and positone problems.

Chapter 3

Preliminaries

This chapter is concerned with the discussion of some concepts and recent results dealing with other interesting and more critical boundary value problems. It gives highlight on the pre-requisite idea to deal with nonlinear analysis, ordinary differential equations or the boundary value problems particularly in the chosen areas such as fixed point theorems, initial value problems, problems on annular domain and positone problems.

3.1 Homeomorphism

Let $M, N \subset \mathbb{R}^n$ be open sets. A function $\phi : M \longrightarrow N$ is called a homeomorphism (sometimes a C^0-diffeomorphism), if it is continuous, bijective and its inverse is also continuous. The function is called a C^k-diffeomorphism, if it is k-times continuously differentiable, bijective and its inverse is also k-times continuously differentiable, simple examples are functions like $\psi, \varphi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\psi(x) = sin(x)$ and $\varphi(x) = \exp(x), \forall x \in \mathbb{R}$.
All these types were considered in ([21],[32]) in the scalar case and for periodic or Neumann problems with a non-linearity defined only on the derivative. Standard examples of classical homeomorphism are $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $\phi(s) = s$ for which the quasilinear second order equation

$$[\phi(u')]' = f(t, u, u') \qquad (3.1.1)$$

is the semi-linear system

$$u'' = f(t, u, u') \tag{3.1.2}$$

or to $\phi(s) = \phi_p(s) := |s|^{p-2}s$, $(p > 1)$,where $\phi(s) = \phi_p(s)$ = classical homeomorphism and ($|.|$ is the euclidean norm in $\mathbb{R}^n$) for (3.1.1) above is the quasilinear system associated to the p-Laplacian

$$\left(|u'|^{p-2}u'\right)' = f(t, u, u'). \tag{3.1.3}$$

An example of a bounded homeomorphism is the map $\phi : \mathbb{R}^n \longrightarrow B(b), (b < +\infty)$ where $B(b)$ is a ball in $\mathbb{R}^n$ centred at b defined by

$$\phi(s) = \phi_c(s) := \frac{s}{\sqrt{1 + |s|^2}}, \forall s \in \mathbb{R}^n,$$

where $\phi(s) = \phi_c(s)$ = bounded homeomorphism and for which (3.1.1) reduces for $n = 1$ to quasilinear equations associated to curvature or capillary problem

$$\left(\frac{u'}{\sqrt{1 + (u')^2}}\right)' = f(t, u, u'). \tag{3.1.4}$$

An example of singular homeomorphism is the map $\phi : B(a) \longrightarrow \mathbb{R}^n$ where $B(a)$ is a ball in $\mathbb{R}^n$ centred at a defined by

$$\phi(s) = \phi_R(s) := \frac{s}{\sqrt{1 - s^2}},$$

where $\phi(s) = \phi_R(s)$ = singular homeomorphism and for which (3.1.1) reduces to quasilinear equations associated to relativistic acceleration

$$\left(\frac{u'}{\sqrt{1 - (u')^2}}\right)' = f(t, u, u'). \tag{3.1.5}$$

Notice that if ϕ is classical, the same is true for ϕ^{-1},if ϕ is bounded, ϕ^{-1}, is singular and if ϕ is singular, ϕ^{-1}, is bounded. In particular $\phi_p^{-1} = \phi_q$,with $\frac{1}{p} + \frac{1}{q} = 1$, and $\phi_c^{-1} = \phi_R$.

3.2 Operator

An Operator T is simply a function mapping a subset of a normed space into another normed space. Let $X, Y \subset \mathbb{R}^n$ be two normed spaces, and let V be a subset of X. An operator $T : V \subset X \longrightarrow Y$ is continuous at $x_0 \in V$ if for every sequence $\{U_n\}_{n=1}$ in V such that $U_n \to x_0$ as $n \to \infty$, we have that $T(U_n) \to T(x_0)$. The operator T is continuous on V if it is continuous at each $x_0 \in V$. T is called a linear operator if for every $\alpha, \beta \in \mathbb{R}$ or any scalar field and $x, y \in V$ we have

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty.$$

Equivalently an operator $T : V \subseteq X \longrightarrow Y$ is said to be continuous at $x_0 \in V$ if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon\theta, x_0) > 0$ such that for all $x \in V$ satisfying $\|x - x_0\| < \delta$, then $\|Tx - Tx_0\| < \epsilon$. A linear operator $T : X \longrightarrow Y$ is called bounded if there exists a constant $k \geq 0$ such that for every $x \in X$, $\|Tx\| \leq k\|x\|$. If T is a bounded linear operator then the number

$$\|T\| = \sup_{\|x\|=1} \|Tx\|$$

is called the norm of T.

Theorem 3.1

A linear operator $T : X \longrightarrow Y$, (with X,Y normed spaces $\mathbb{R}^n$) is continuous on X if and only if it is bounded.

Proof.

Sufficiency: From the inequality

$$\|Tx\| \leq k\|x\|, \qquad \forall x \in X. \tag{3.2.1}$$

It follows that

$$\|Tx - Tx_0\| \leq K\|x - x_0\|, \tag{3.2.2}$$

for any $x_0, x \in X$. Thus, if $U_n \to x_0$, then $TU_n \to Tx_0$.

Necessity: Suppose that T is continuous on X. We show that

$$k_0 = \sup_{\|x\|=1} \|Tx\| < +\infty. \tag{3.2.3}$$

In fact, let $k_0 = +\infty$. Then there exists a sequence $\{U_n\} \subset X$ such that $|U_n| = 1$ and $|TU_n| \longrightarrow \infty$. Let $\lambda_n = |TU_n|$. We may assume that $\lambda_n > 0$ for all n. Let $\bar{U_n} = \frac{U_n}{\lambda_n}, \ \forall \ n \in N$. Then $\|\bar{U_n}\| = (\frac{1}{\lambda_n})\|U_n\| \longrightarrow 0$. This implies that $\|T\bar{U_n}\| = 1$; a contradiction to the continuity of T. Therefore, $k_0 < +\infty$. Let $x \neq 0$ be a vector in X. Then $\bar{x} = \frac{x}{\|x\|}$ satisfies $\|\bar{x}\| = 1$. Thus $\|T\bar{x}\| = \frac{\|Tx\|}{\|x\|}$ and $\|T\bar{x}\| \leqslant k_0$. Consequently

$$\|Tx\| \leq k_0\|x\| \tag{3.2.4}$$

since (3.2.2) holds also for $x = 0$, we have shown (3.2.1) with $k = k_0$.

Completely Continuous Operator

Let E and X be Banach spaces and let Ω be an open subset of E, let

$$T : \Omega \longrightarrow X$$

be an operator, then T is called compact whenever $T(\Omega')$ is pre-compact in X for every bounded subset Ω' of Ω (ie $T(\Omega')$ is compact in X). We call T completely continuous whenever T is compact and continuous. We note that if T is linear and compact, then T is completely continuous.

Lemma 3.1

Let Ω be an open set in E and let $T : \Omega \longrightarrow X$ be completely continuous, let T be k-differentiable at a point $x_0 \in \Omega$. Then the linear mapping $F = DT(x_0)$ is compact, hence completely continuous.

Proof

Since F is linear it suffices to show that $F(\{x : \|x\| \leq 1\})$ is pre-compact in X since it is a bounded subset of X. (We again shall use the symbol $\|.\|$ to denote the norm in E

and in X). If this were not the case, there exist $\epsilon > 0$ and a sequence $\{U_n\}_{n=1}^{\infty} \subset E$, $\|U_n\| \leq 1, \quad n = 1, 2, ...$ such that $\|FU_n - FU_m\| \geq \epsilon$, $n \neq m$.

Since T is linear, continuous and Frechet differentiable we Choose $0 < \delta < 1$ such that $\|T(x_0 + h) - T(x_0) - F(h)\| < \frac{\epsilon}{3}\|h\|$, for $h \in E$ such that $\|h\| \leq \delta$. Then for $n \neq m$,

$$\|T(x_0 + \delta U_n) - T(x_0 + \delta U_m)\| \geq \delta \|FU_n - FU_m\| - \|T(x_0 + \delta U_n) - T(x_0) - \delta FU_n\|$$
$$-\|T(x_0 + \delta U_m) - T(x_0) - \delta FU_m\| \geq \delta\epsilon - \frac{\delta\epsilon}{3} - \frac{\delta\epsilon}{3} = \frac{\delta\epsilon}{3}.$$

Hence, the sequence $\{T(x_0 + \delta U_n)\}_{n=1}^{\infty}$ has no convergent subsequence. On the other hand for $\delta > 0$, small, the sequence $\{x_0 + \delta U_n\}_{n=1}^{\infty} \subset \Omega$, and is bounded, implying by the complete continuity of T that $\{T(x_0 + \delta U_n)\}_{n=1}^{\infty}$ is pre-compact, a contradiction.

3.3 Boundary Value Problems

We have chosen to consider the case of two-point second order boundary value problems

$$U'' + \lambda U = 0, U(0) = 0 \;\; and \;\; U(L) = 0. \tag{3.3.1}$$

Our goal is to find the eigenvalue λ such that the boundary value problem (3.3.1) will have a nonzero solution satisfying both boundary conditions. Since the form of the general solution of the second order linear equation is dependent on the type of roots that its characteristic equation has. In this example the characteristic equation is

$$r^2 + \lambda = 0. \tag{3.3.2}$$

We observe that the type of roots (3.3.2) has is dependent on its discriminant, which is simply -4λ. We will attempt to find λ by considering the 3 possible types of the solution arising from the different roots of the characteristic equation.

Case 1: If $\lambda < 0$. Now, denote $\lambda = -\sigma^2$, where $\sigma = \sqrt{-\lambda} > 0$, then (3.3.2) becomes $r^2 + \lambda = r^2 - \sigma^2 = 0$, which now has roots $r = \pm\sigma$. The general solution is then

$$U(x) = c_1 \exp(\sigma x) + c_2 \exp(-\sigma x). \tag{3.3.3}$$

Applying the boundary conditions, we have $U(0) = 0 = c_1 + c_2 \Rightarrow c_2 = -c_1$. and

$$U(L) = 0 = c_1 \exp(\sigma L) + c_2 \exp(-\sigma L) = c_1(\exp(\sigma L) - \exp(-\sigma L))$$

So either c_1 (which implies $c_2 = 0$,since $c_2 = -c_1$) or $\exp(\sigma L) - \exp(-\sigma L) = 0$.

Now $c_1 = c_2 = 0$ implies from (3.3.3) that $u(x) = 0$, a trivial solution which does not lead to the general solution later on.

On the other hand $\exp(\sigma L) - \exp(-\sigma L) = 0$. leads to no solution of the system (3.3.2). Thus, there is no negative eigenvalue for this problem.

Case 2: If $\lambda = 0$. The equation (3.3.1) becomes $U'' = 0$ and has the general solution as

$$U(x) = c_1 + c_2 x. \tag{3.3.4}$$

Applying the boundary conditions we get $U(0) = 0 = c_1 \quad \Rightarrow c_1 = 0$.

$$U(L) = 0 = c_1 + c_2 L = c_2 L \quad \Rightarrow c_2 = 0, (\text{ since L} > 0).$$

Hence, $U(x) = 0 + 0x = 0$, the trivial solution is the only possibility. Therefore, zero is not an eigenvalue for this problem.

Case 3: If $\lambda > 0$ $(-4\lambda < 0$, complex roots of (3.3.2). Let us denote $\lambda = \sigma^2$, where $\sigma = \sqrt{\lambda} > 0$. The characteristic equation becomes

$$r^2 + \lambda = r^2 + \sigma^2 = 0$$

and has roots $r = \pm\sigma i$. The general solution then is

$$U(x) = c_1 cos(\sigma x) + c_1 sin(\sigma x). \tag{3.3.5}$$

Applying the boundary conditions, we get $U(0) = 0 = c_1 cos(0) + c_2 sin(0) = c_1 \Rightarrow c_1 = 0$ and

$$U(L) = 0 = c_1 cos(\sigma L) + c_2 sin(\sigma L) = c_2 sin(\sigma L).$$

Observe that this last equation has 2 possible solutions: either

$c_2 = 0$ $(= c_1)$, (which results in the trivial solution again) or, it could be that $sin(\sigma L) = 0$, which means $\sigma L = \pi, 2\pi, 3\pi, ..., n\pi$ (or $\sigma L = n\pi,\ n \geq 1$). That is, there are infinitely many values $\sigma = \dfrac{n\pi}{L},\ n \geq 1$, such that there exists a nonzero solution of this boundary value problem. The positive values of λ for which the equation will have a solution satisfying the specified boundary condition is the eigenvalues of this boundary value problem, i.e;

$$\lambda = \sigma^2 = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, ... \tag{3.3.6}$$

which are all the positive integers. In the general solution (3.3.5) $c_1 = 0$ and c_2 could be any nonzero constant. Therefore, the eigenfunctions corresponding to the eigenvalues found above which are the actual nonzero solutions that satisfy the given set of boundary conditions when the original differential equation has $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ as its coefficient are

$$U_n = sin\frac{n\pi x}{L}, \quad n = 1, 2, 3.... \tag{3.3.7}$$

In a more general context, let J be a subinterval of $\mathbb{R}$ and $F : J \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuous. Let B_J be the class of continuous $\mathbb{R}^n$-valued functions defined on J. Then the system

$$U' = F(t, x) \tag{3.3.8}$$

or $U' = A(t)U$ or more generally $U' = A(t)U + f(t), f : J \longrightarrow \mathbb{R}^n$, along with the condition $x \in B_J$ is a boundary value problem on the interval J. Naturally, the condition $x \in B_J$ is too general and contains the initial value problem on $J = [t_0, T]$, that is B_J can be the class $\{x \in C_n[t_0, T] : u(t_0) = x_0\}$. A boundary value problem usually concerns itself with boundary conditions of the form $x \in B_J$ which involve values of the function x at more than one point of the interval J. One of the most important boundary value problems in the theory of ordinary differential equations is the problem concerning the existence of a T- periodic solution. This problem consist of (3.3.8) and the condition

$$u(t + T) = u(t), \quad t \in \mathbb{R} \tag{3.3.9}$$

where T is a fixed positive number. Here it is usually assume that F is T - periodic in its first variable t as in (3.3.9) which ranges over $\mathbb{R}$. In this case, this problem is actually reduced to the simple problem

$$u(0) - u(T) = 0. \tag{3.3.10}$$

Another general boundary value problem is the problem (3.3.8) and

$$Mu(0) - Nu(T) = 0 \tag{3.3.11}$$

or

$$U'' + \lambda U = 0, U(0) = 0, U(2\pi) = 0.$$

Here M and N are known $n \times n$ matrices. Naturally, the points $0, T$ can be replaced above and in what follows, by any other points $a, b \in \mathbb{R}$ with $b > a$.

In the case $M = N = I$, the condition (3.3.11) coincide with (3.3.10). An even more general problem is the problem (3.3.9) and

$$\mu(x) = r \tag{3.3.12}$$

Here $r \in \mathbb{R}^n$ is fixed and μ is a linear operator with domain in $C_n[0, T]$ and values in $\mathbb{R}^n$. Such an operator μ could be given by

$$\mu(x) = \int_0^T \vartheta(s)u(s)ds \tag{3.3.13}$$

where $\vartheta : [0, T] \longrightarrow M_n$ is continuous. The operator μ in (3.3.12) could be defined on a class of functions over an infinite interval. e.g

$$\mu(x) = \int_0^\infty \vartheta(s)u(s)ds \tag{3.3.14}$$

or

$$\mu(x) = Mu(0) - Nu(\infty) \tag{3.3.15}$$

where $x(\infty)$ denote the limit of $x(t)$ at $t \to \infty$. In these last two conditions we may consider

$$B_J = \{\mu \in C'_n : \mu_u = r\}.$$

Naturally, we must also assume a condition on ϑ like

$$\int_0^\infty \|\vartheta\,(t)\,\|dt < +\infty.$$

3.4 Leray-Schauder Fixed Point Theorem

A large number of problems in the field of differential equations can be reduced to the problem of finding a solution x of an equation of the form $Tx = y$, where T is an operator that maps a subset of a Banach space X into some other Banach space Y and y is a known element of Y. If $y = 0$ and $Tx = Ux - x$, for some other operator U, then the equation $Tx = y$ is equivalent to the equation $Ux = x$. Naturally in order to solve $Ux = x$, we must assume that the domain $D(U)$ and the range $R(U)$ have points in common. Point x for which $Ux = x$ are called the fixed point of the operator U.

The fixed point theorems which are most widely used in differential equation are the Banach contraction principle, the Schauder-Tychonov theorem and the Leray-Schauder theorem . But of particular interest is the Leray-Schauder fixed point theorem below:

Theorem 3.2 (Leray-Schauder)

Let X be a Banach space and consider the operator $S : [0,1] \times X \longrightarrow X$ and the equation

$$x - S(t,x) = 0 \tag{3.4.1}$$

under the following hypothesis

(1) $(t, .)$ is compact for all $t \in [0,1]$. Moreover, for every bounded set $M \subset X$ and every $\epsilon > 0$ there exist $\delta(\epsilon, M) > 0$ such that $\|S(t_1,x) - S(t_2,x)\| < \epsilon$ for every $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta(\epsilon, M)$ and every $x \in M$;

(2) $S(t_0, x) = 0$ for some $t_0 \in [0,1]$ and every $x \in X$;

(3) there exists a constant $k > 0$ such that $\|x_t\| \leq k$ for every solution x_t of (3.4.1).

Then equation (3.4.1) has a solution for every $t \in [0,1]$. S is called the homotopy of compact operator.

3.5 Quasilinear Systems

The most important property of a quasilinear system

$$U' = A(t,x)x + F(t,x) \tag{3.5.1}$$

with $A : J \times \mathbb{R}^n \longrightarrow M_n, F : J \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ (with J a real interval), is that the system

$$U' = A\left(t, f(t)\right) x + F\left(t, f(t)\right) \tag{3.5.2}$$

is linear for any $\mathbb{R}^n$ - valued function f on J. Linear systems of the type (3.5.2) have been known and it is therefore natural to ask whether information about (3.5.1) can be obtained by somehow exploiting the properties of the system (3.5.2), where f belongs to a certain class $A(J)$ of continuous functions on J.

It is shown here that some of the properties of the system (3.5.2) can be carried over to the system (3.5.2) via fixed point theory. In fact, if U denotes the operator which maps the function $f \in A(J)$ into the unique solution $x_f \in A(J)$ of (3.5.2), then the fixed points of U are solutions in $A(J)$ of the system (3.5.1). This procedure is followed here in order to obtain some stability and periodicity properties of the system (3.5.1).

It should be noted that the quasilinear system constitute quite a large class. To see this it suffices to observe that if $B : J \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuously differentiable with respect to its second variable, then there exists a matrix $A(t, x)$ such that

$$B\left(t, x\right) \equiv A\left(t, x\right) x + B\left(t, 0\right), \quad (t, x) \in J \times \mathbb{R}^n. \tag{3.5.3}$$

This assertion follows from the following lemma:

Lemma 3.2

Let D be an open convex subset of $\mathbb{R}^n$. Let $F : D \longrightarrow \mathbb{R}^n$ be continuously differentiable on D. Let $x_1, x_2 \in D$ be given. Then for $s \in [0, 1]$,

$$F\left(x_2\right) - F\left(x_1\right) = \int_0^1 F_x\left(Sx_2 + (1 + s)\, x_1\right) ds \tag{3.5.4}$$

where $F_x(u)$ is the Jacobian matrix $\left[\frac{\partial F_i(u)}{\partial x_j}\right], i, j = 1, 2, 3, ...n$ of F at u.

Proof:

Consider the function

$$g(s) = F\left(sx_2 + (1 - s)\, x_1\right), s \in [0, 1]. \tag{3.5.5}$$

This function is well define because the set D is convex. Using the chain rule for vector-valued functions, we have

$$g'(s) = F_x\left(sx_2 + (1-s)\,x_1\right)(x_2 - x_1)\,. \tag{3.5.6}$$

Integrating (3.5.6) from $s = 0$ to $s = 1$ and recalling that $g(0) = F(x_1)$, $g(1) = F(x_2)$ we get (3.5.4) above.

3.5.1 Quasilinear Equation Of Second Order

Here, we consider the equation

$$\sum_{i,j=1}^{n} a^{ij}\left(x, u, \nabla u\right) u_{x_i x_j} + b\left(x, u, \nabla u\right) = 0 \tag{3.5.7}$$

in a domain $\Omega \subset \mathbb{R}^n$, where $U : \Omega \longrightarrow \mathbb{R}$ and $a^{ij} = a^{ji}$ with a characteristic equation

$$\sum_{i,j=1}^{n} a^{ij}\left(x, u, \nabla u\right) \chi_{x_i} \chi_{x_j} = 0$$

and in contrast to linear equations the solution of the characteristic equation depends on the solution considered. There are large class of quasilinear equations such that the associated characteristic equation has no solution χ, $\nabla\chi \neq 0$. Now, set $U = \{(x, z, p) : x \in \Omega, z \in \mathbb{R}, p \in \mathbb{R}^n\}$

Definition

The quasilinear equation (3.5.9) is called elliptic if the matrix $\left(a^{ij}\left(x, z, p\right)\right)$ is positive definite for each $(x, z, p) \in U$.

Assume equation (3.5.9) is elliptic and let $\lambda\left(x, z, p\right)$ be the minimum and $\Lambda\left(x, z, p\right)$ the maximum of the eigenvalues of (a^{ij}), then

$0 < \lambda\left(x, z, p\right) |\epsilon|^2 \leq \sum_{i,j=1}^{n} a^{ij}\left(x, z, p\right) \zeta_i \zeta_j \leq \Lambda\left(x, z, p\right) |\zeta|^2 \quad$ for all $\quad \zeta \in \mathbb{R}^n$.

Definition

Equation (3.5.9) is called uniformly elliptic if $\frac{\Lambda}{\lambda}$ is uniformly bounded in U. An important class of elliptic equation which are not uniformly elliptic (non-uniformly elliptic) is

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{U_{x_i}}{\sqrt{1 + |\nabla u|^2}}\right) + \text{lower order terms } = 0. \tag{3.5.8}$$

In general, the behaviour of uniformly elliptic equations is similar to linear elliptic equation in contrast to the behaviour of solution of non-uniformly elliptic equation ([36]).

Systems Of Second Order Quasilinear Differential Equation

In this section we consider the system

$$\sum_{k,l=1}^{n} A^{kl}\left(x, u\nabla u\right) U_{x_k,x_l} + \text{lower order term } = 0, \tag{3.5.9}$$

where A^{kl} are $M \times M$ matrices and $u = (u_1, u_2, u_3, ..., U_n)^T$. We assume $A^{kl} = A^{lk}$, with no restriction of generality provided $U \in C^{\Omega}$ is satisfied. As earlier discussed, the classification follows from the question whether or not we can calculate formally the solution from the differential equations if sufficiently many data are given on an interval manifold ([36]).

Chapter 4

Main Result

4.1 Introduction

In this project, we are mostly concerned with boundary value problems for non-linear partial differential equations.The types of non-linear partial differential equations we have in mind arise in most areas of applied sciences such as in the study of elasticity theory, astrophysics, porous media and plasma problems ([15],[27],[43],[30], [51]).
Let

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad N \geq 1,$$

be a continuous mapping that behaves asymptotically like

$$A(v).v \approx |v|^p,$$

where $p \in (1, \infty)$, and more specifically

$$A(v).v \geq \alpha |v|^p,$$

and $|A(v).v| \leq \beta |v|^{p-1}$, where α and β are positive constants.The function A is assumed to be strictly monotone with respect to v i.e

$$(A(v_1) - A(v_2)).(v_1 - v_2) > 0, \quad v_1 \neq v_2.$$

Now, we consider the boundary value problem

$$-divA(\nabla u(x)) = f(\lambda, u(x)), \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega, \tag{4.1.1}$$

where

$\Omega =$ bounded domain in $\mathbb{R}^n$,

$\partial\Omega =$ boundary of Ω

$\nabla u(x) = (u_{x_1}, u_{x_2}, u_{x_3}..., u_{U_n})$

$f =$ a Lipschitz function.

The special case of equation (4.1.1) is when the operator A has the form

$$A(\nabla u(x)) = a(|\nabla u(x)|)\nabla u(x),$$

and when Ω is an annular domain in $\mathbb{R}^n$ (i.e Ω is either a ball or a spherical shell). In these special cases i.e a quasilinear second order differential systems of the form

$$[\phi(u')]' = f(t, u, u'), \tag{4.1.2}$$

where $f : [0, t] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous, $\phi : B(a) \longrightarrow B(b)$ belongs to a suitable class of homeomorphism with $B(\rho) \subset \mathbb{R}^n$ the open ball of centre 0 and radius ρ, $B(+\infty) = \mathbb{R}^n$, $0 < a, b \leq +\infty$. It is of interest to seek solutions u which only depend on the radial variables, i.e u such that $u(x) = u(|x|) = u(r)$. A solution of (4.1.2) on $[0, t]$ is a function $U \in C'([0, t], \mathbb{R}^n)$ such that $U'(t) \in B(a)$ for all $t_* \in [0, t]$, $\phi \circ U \in C'([0, t], \ \mathbb{R}^n)$. Such solutions are then solutions of the boundary value problems for ordinary differential equations

$$\left[r^{N-1}\phi(u^{'})\right]^{'} + r^{N-1}g(\lambda, u) = 0, \quad r \in (0, R), \quad u^{'} = 0, \quad u(R) = 0 \tag{4.1.3}$$

(in case Ω is a ball of radius $R \in \mathbb{R}^n$) and

$$\left[r^{N-1}\phi(u^{'})\right]^{'} + r^{N-1}g(\lambda, u) = 0, \quad r \in (a, b), \quad u(a) = u(b) = 0 \tag{4.1.4}$$

(in case $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\},$) where ϕ is an increasing homeomorphism of $\mathbb{R}$, with $\phi(0) = 0$ and $g \in C(R)$ such that $g(x) \geq 0$ for all $x > 0$ and $g(0) = 0$. The class of homeomorphism ϕ occuring in the equation above is characterized by the following conditions. ϕ is a homomorphism from $B(a) \in \mathbb{R}^n$ onto $\mathbb{R}^n$ such that $\phi(0) = 0$, $\phi = \nabla\Phi$

with $\Phi : \bar{B(a)} \longrightarrow \mathbb{R}^n$ of class C' on $B(a)$, continuous strictly convex on $\bar{B(a)}$, and such that $\Phi(0) = 0$. So ϕ is strictly monotone on $B(a)$, in the sense that
$\langle \phi(u) - \phi(v), \quad u - v\rangle > 0$ for $u \neq v$, and Φ reaches its minimum 0 at 0 ([21]). Examples of these categories are given in section (3.1) and we shall study these classes of boundary value problems for various categories of ϕ and g. (It will be assumed that $ug(\lambda, u) \quad \geq 0$). In a more general and convenient form (4.1.3) and (4.1.4) can be written as:

$$\Big[\phi(u^{'})\Big]^{'} + \frac{N-1}{r}\phi(u^{'}) + g(\lambda, u) = 0. \tag{4.1.5}$$

4.2 Tools Of Analysis And Organization

This project is an appraisal and reports about some recent papers, where tools from nonlinear analysis have been used to analyse boundary value problems of type (4.1.3) and (4.1.4) and of particular reference are ([35],[26],[25],[24],[22],[17],[18],[14],[4]). The tools employed are the nonlinear functional analysis from the work of ([5],[6],[7],[29],[42],[44],[50]). On the organization, the first part is concerned with problem (4.1.3) where we consider the case of nonlinearities of g which grow at the same rate as the nonlinear term ϕ and are linear with respect to the parameter λ. We then continue with problems were g grows superlinearly (but subcritically) with respect to ϕ. In the other part we consider the boundary value problems on an annular domain as given in problem (4.1.4), where the term g grow superlinearly with respect to ϕ.

4.3 Boundary Value Problems On a Ball

4.3.1 Equivalent Integral Equation

In ([36]) it was defined that an integral equation of a characteristic system is a function $\phi(x, y)$ where ϕ is an odd increasing homeomorphism on $\mathbb{R}$, λ is a positive parameter (i.e we consider the case that $\alpha = \gamma = N - 1$) and

$$\phi(x(t), y(t)) = k,$$

(k = constant) for each characteristic curve. The constant depends on the characteristic curve considered ([31]). In this section we shall derive an integral equation whose solution set is the set of positive solution of

$$\left[r^{N-1}\phi(u^{'})\right]^{'} + r^{N-1}g(\lambda,u) = 0, \quad r \in (0,R), \quad u^{'}(0) = 0, \quad u(R) = 0. \qquad (**)$$

Fundamentally, finding positive solutions of (**) is equivalent to finding nontrivial solutions of the problem

$$\left[r^{N-1}\phi(u^{'})\right]^{'} + r^{N-1}g(\lambda,|u|) = 0, r \in (0,R), u^{'}(0) = 0, u(R) = 0. \qquad (4.3.1)$$

To achieve this derivation let

$$C_{\sharp} = \{u \in C[0,R] : u(R) = 0\};$$

denote a closed subspace of $C[0,R]$. Then $C_{\sharp}$ is a Banach space for the norm $|\cdot| = |\cdot|_{\infty}$. Suppose u is a solution of (4.3.1) then integrating (4.3.1) from r to R, we obtain

$$\int_r^R [\xi^{N-1}\phi(u^{'}(\xi))]^{'} d\xi + \int_r^R \xi^{N-1} g(\lambda,|u(\xi)|)d\xi = 0.$$

This implies that

$$\xi^{N-1}\phi(u^{'}(\xi))|_r^R + \int_r^R \xi^{N-1} g(\lambda,|u(\xi)|)d\xi = 0.$$

Thus,

$$\xi^{N-1}\phi(u^{'}(R)) - \xi^{N-1}\phi(u^{'}(r)) = -\int_r^R \xi^{N-1} g(\lambda,|u(\xi)|)d\xi$$

Since $u'(0) = u(R) = 0$ by hypothesis, then on evaluation and further integration we have

$$u(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1} g(\lambda,|u(\xi)|)d\xi\right] ds \qquad (4.3.2)$$

Now, let us define an operator $T : C_{\sharp} \times [0,+\infty] \longrightarrow C_{\sharp}$ by

$$T(u,\lambda)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1} g(\lambda,|u(\xi)|)d\xi\right] ds, \quad u \in C_{\sharp}, \quad 0 \le \lambda \le \infty. \quad (4.3.3)$$

T is well defined and linear. To see this let $(u_1,\lambda_1), \quad (u_2,\lambda_2) \in C_{\sharp} \times (0,\infty)$, then

$$T((u_1,\lambda_1) + (u_2,\lambda_2))(r) = T(u_1 + (u_2,\lambda_1) + \lambda_2)(r)$$

$$= \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}g(\lambda_1) + \lambda_2, |u_1 + (u_2(\xi)|)d\xi\right] ds$$

$$= \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}g((\lambda_1, |u_1(\xi)|) + (\lambda_2, |u_2(\xi)|))d\xi\right] ds$$

$$= \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}g((\lambda_1, |u_1(\xi)|)d\xi\right] ds + \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}g(\lambda_2, |u_2(\xi)|)d\xi\right] ds$$

$$= T(u_1, \lambda_1)(r) + T(u_2, \lambda_2)(r).$$

Also

$$T(\alpha u, \lambda)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}g(\lambda, |\alpha u(\xi)|)d\xi\right] ds$$

$$= |\alpha| \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}g(\lambda, |u(\xi)|)d\xi\right] ds$$

$$= \alpha T(u, \lambda)(r).$$

Therefore, T is well defined and the fixed point $T(\cdot, \lambda)$ will provide the solution of

$$[r^{N-1}\phi(u^{'})]' + r^{N-1}g(\lambda, |u|) = 0, \quad r \in (0, R), \quad u^{'}(0) = 0, \quad u(R) = 0.$$

4.3.2 Eigenvalue Problems

This section is intended to consider the problem

$$\left[r^{N-1}\phi(u^{'})\right]^{'} + \lambda r^{N-1}\psi(u) = 0, r \in (0, R), \quad u^{'}(0) = 0, \quad u(R) = 0 \tag{4.3.4}$$

where ϕ is an increasing homeomorphism of $\mathbb{R}$ and ψ is a nondecreasing function with $\phi(0) = 0, \quad \psi(0) = 0$, and for any $\sigma > 0$

$$A(\sigma) \leq \frac{\phi(\sigma|x|)}{|\psi(x)|} \leq \gamma(\sigma), \tag{4.3.5}$$

where $A(\sigma)$ and $\gamma(\sigma)$ are positive constants depending on σ only.This problem is a natural extension of the eigenvalue problem for the p-Laplacian ($\phi(u) = \psi(u) = |u|^{p-2}u, \quad p > 1$) considered by ([3],[10],[40]).

Next, we determine the existence of the value of λ such that:

(1) the problem (4.3.4) has positive solutions. These values of λ shall be called the principal eigenvalues and

(2) the problem has a nontrivial sign changing solution and these values shall be called the higher eigenvalues ([41],[49],[28]).

4.3.3 On The Principal Eigenvalues

Definition

Let X be a Banach space and let

$$F : X \times I \longrightarrow \mathbb{R}$$

be a completely continuous operator on $X \times I$, where I is a real interval. Now, consider the equation

$$F(x, \lambda) = x.$$

Suppose that $F(0, \lambda) = 0, \forall \lambda \in I$ and that $\bar{\lambda} \in I$, then we say that $(0, \bar{\lambda})$ is a bifurcation point of the equation

$$F(0, \lambda) = x,$$

near zero if in any neighbourhood of $(0, \bar{\lambda})$ there exist a nontrivial solution of the equation or equivalently there is a sequence U_n in X and λ_n in I with $(U_n, \lambda_n) \to (0, \bar{\lambda})$ in $X \times I$, then $[U_n, \lambda_n]$ is a solution of the equation $F(x, \lambda) = x$, $\forall n \in N$.

Now, let us denote

$$S = \{(\lambda, u) \in R_+ \times C_\sharp : (\lambda, u) \text{ is a solution of } (4.3.4) \text{ , u(r)} > 0, \text{ r} \in (0, \text{R})\} \quad (4.3.6)$$

Then, we establish the following theorem.

Theorem 4.1

Let S be as defined above in (4.3.6) and let ϕ and ψ be homeomorphisms such that for any $\sigma > 0$

$$A(\sigma) \leq \frac{\phi(\sigma|x|)}{|\psi(x)|} \leq \gamma(\sigma).$$

Then S is not empty ($S \neq \emptyset$) and there exist numbers $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $(\lambda, u) \in S$ implies $\lambda_1 \leq \lambda \leq \lambda_2$. Also there exists a continuum $C \subset \bar{S}$ which is unbounded in $[\lambda_1, \lambda_2] \times C_\sharp$ and bifurcates from $[\lambda_1, \lambda_2] \times 0$.

Proof

Let us consider the equivalent operator equation. The operator T in this case has the form

$$T(u,\lambda)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\lambda\psi(|u(\xi)|)d\xi\right] ds. \tag{4.3.7}$$

This project now attempted to show that

$$deg_{LS}(I - T(\cdot,0), B(0,R_1), 0) = 1 \tag{4.3.8}$$

and

$$deg_{LS}(I - T(\cdot,\lambda_2), B(0,R_1), 0) = 0, \tag{4.3.9}$$

for some $\lambda_2 > 0$ and $R_1 > 0$, where deg_{LS} implies Leray - Schauder degree. That (4.3.8) holds is obvious, since $T(\cdot,0) = 0$. To show that (4.3.9) holds; we consider the operator $T_\epsilon : C_\sharp \times R_+ = \quad (0,\infty) \longrightarrow C_\sharp$, defined by

$$T_\epsilon(u,\lambda)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\lambda(\phi(|u(\xi)|)+\epsilon)d\xi\right] ds, \tag{4.3.10}$$

where $\epsilon > 0$ is a constant, T_ϵ is a completely continuous operator that maps bounded sets of $C_\sharp \times (0,\infty)$ into bounded set of $C_\sharp$ and $T_\epsilon(\cdot,0) = 0$. Furthermore, whenever $deg_{LS}(I - T(\cdot\lambda),\quad B(0,R_1),0)$ is defined then

$$deg_{LS}(I - T(\cdot,\lambda),\quad B(0,R_1),0) = deg_{LS}(I - T_\epsilon(\cdot,\lambda),\quad B(0,R_1),0) \tag{4.3.11}$$

for all small ϵ.

Equation (4.3.9) will hold if we can show that there exists $\bar{\lambda}$ such that (u,λ) is a solution of $T_\epsilon(\lambda,u) = u$, implies $\lambda \le \bar{\lambda}$ and that this number is independent of ϵ for $0 \le \epsilon \le \epsilon_0$.

Now, suppose that the problem

$$\left[r^{N-1}\phi(u')\right]' + \lambda r^{N-1}\psi(|u(r)|) + \epsilon = 0$$

is given then

$$r^{N-1}\phi(u')|_r^R + \int_0^r (\lambda\xi^{N-1}\psi(|u(\xi)|) + \epsilon)d\xi = 0$$

and

$$-r^{N-1}\phi(u^{'}) = -\int_0^r (\lambda\xi^{N-1}\psi(|u(\xi)|) + \epsilon)d\xi \geq 0$$

is satisfied by u and

$$u(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(|u(\xi)|) + \epsilon)d\xi\right] ds \geq 0.$$

$\forall\ r \in [0, R]$. Also for $r \in [\frac{R}{4}, \frac{3R}{4}] \subset [0, R]$

$$u(r) \geq \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \lambda\xi^{N-1}\psi(|u(\xi)|)d\xi\right] ds.$$

Thus, for all $r \in \left[\frac{R}{4}, \frac{3R}{4}\right]$, we have that

$$u(r) \geq \int_r^R \phi^{-1}\left[\lambda\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\psi(|u(\xi)|)d\xi\right] ds$$

$$\Rightarrow u(r) \geq \int_r^R \phi^{-1}\left[\lambda\frac{1}{s^{N-1}}\psi(u(r))\int_0^s \xi^{N-1}d\xi\right] ds$$

$$u(r) \geq \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\lambda\psi(u(r)) \times \frac{\xi^N}{N}|_0^{\frac{3s}{4}}\right] ds$$

$$= \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\frac{\lambda}{N}\psi(u(r))\left((\frac{3s}{4})^N - 0\right)\right] ds = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\frac{\lambda}{N}\psi(u(r))\frac{3^N s^N}{4^N}\right] ds$$

$$= \int_r^R \phi^{-1}\left[\frac{\lambda}{N}\psi(u(r))\frac{3^N s}{4^N}\right] ds = \phi^{-1}\frac{\lambda}{N}\psi(u(r))\frac{3^N s^2}{4^N 2}|_{\frac{R}{4}}^{\frac{3R}{4}}$$

$$= \phi^{-1}\frac{\lambda}{N}\psi(u(r))\frac{3^N}{4^N}\left[\frac{(\frac{3R}{4})^2}{2} - \frac{(\frac{R}{4})^2}{2}\right] = \phi^{-1}\frac{\lambda}{N}\psi(u(r))\frac{3^N}{4^N}\left[\frac{R^2}{4}\right]$$

$$u(r) \geq \phi^{-1}\frac{R}{4}\left[\frac{R\lambda 3^N}{N4^N}\psi(u(r))\right]$$

$$u(r) \geq \phi^{-1}\frac{R}{4}\left[\frac{R\lambda}{N4^N 3^{-N}}\psi(u(r))\right]$$

$$\Longrightarrow \frac{\phi\left(\frac{4}{R}u(r)\right)}{\psi(u(r))} \geq \frac{R\lambda}{N4^N 3^{-N}}. \tag{4.3.12}$$

But this result cannot hold for λ large, as follows from condition (4.3.5), and S as defined in (4.3.6) is not empty ($S \neq \emptyset$) and λ_2 exists (by using the homotopy invariance of Leray-Schauder degree). The existence of an unbounded continuum as claimed, follows from global Krasnoselskii-Rabinowitz bifurcation theorem in ([9],[13],[34]). To complete the proof,we need to show the existence of λ_1 . Thus let u be a solution with $u(0) = d$. It follows that given

$$u(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(|u(r)|))d\xi\right]ds,$$

then for $r = 0$, we have

$$u(0) = \int_0^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \lambda\xi^{N-1}\psi(|u(0)|)d\xi\right]ds$$

$$d = \int_0^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(d)d\xi\right]ds$$

$$d = \int_0^R \phi^{-1}\left[\frac{1}{s^{N-1}}\lambda\psi(d)\frac{s^N}{N}\right]ds$$

$$d = \int_0^R \phi^{-1}\left[\lambda\psi(d)\times\frac{s}{N}\right]ds \tag{4.3.13}$$

$$d = \phi^{-1}\lambda\psi(d)\times\frac{s^2}{2N}|_0^R$$

$$\Longrightarrow d = \phi^{-1}\lambda\psi(d)\times\frac{R^2}{2N}$$

$$\Longrightarrow d \leq \phi^{-1}\lambda\psi(d)\frac{R^2}{2N}$$

$$\Longrightarrow \phi(\frac{d}{R}) \leq \lambda\psi(d)\frac{R}{N}$$

$$\Longrightarrow \frac{\phi(\frac{d}{R})}{\psi(d)} \leq \frac{R\lambda}{N}$$

Corollary 4.1

Let (u, λ) be a solution of

$$\left[r^{N-1}\phi(u^{'})\right]^{'} + \lambda r^{N-1}\psi(u) = 0,\ r \in [0, R],\ u^{'}(0) = 0,\ u(R) = 0,$$

with $u(0) = d$ and let $\theta \in (0, 1)$ be fixed. Let $r_0 \in (0, R)$ be such that $u(r_0) = d\theta$. Then

$$r_0 \geq \frac{N}{\lambda}A\left(\frac{1-\theta}{R}\right) \tag{4.3.14}$$

Proof

Using the earlier result,we obtain that

$$u(r_0) = \int_{r_0}^{R} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(|u(\xi)|))d\xi\right] ds,$$

but $u(r_0) = d\theta$

$$\therefore \quad d\theta = \int_{r_0}^{R} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(|u(\xi)|))d\xi\right] ds \tag{4.3.15}$$

and

$u(0) - u(r_0) = (1-\theta)d$

$$= \int_{0}^{R} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(|u(\xi)|))d\xi\right] ds - \int_{r_0}^{R} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(|u(\xi)|))d\xi\right] ds$$

$$= \int_{0}^{r_0} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(|u(\xi)|))d\xi\right] ds + \int_{r_0}^{R} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(|u(\xi)|))d\xi\right] ds -$$

$$\int_{r_0}^{R} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(|u(\xi)|))d\xi\right] ds$$

$$\Longrightarrow (1-\theta)d \le \int_{0}^{r_0} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(|u(\xi)|))d\xi\right] ds$$

Let $|u(\xi)| = d$

$$\therefore (1-\theta)d \le \int_{0}^{r_0} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(d)d\xi\right] ds,$$

and

$$(1-\theta)d \le \int_0^{r_0} \phi^{-1}\lambda\frac{d\psi}{N}sds \le \phi^{-1}\lambda\psi(d)\frac{r_0^2}{2N} \le R\phi^{-1}\left[\frac{\lambda\psi(d)r_0}{N}\right],$$

where $R = \frac{r_0}{2}$, which is the conclusion. This result has the following consequence.

Corollary 4.2

Let $\{(U_n, \lambda_n)\}$ be a sequence of solutions of the problem

$$\left[r^{N-1}\phi(u^{'})\right]^{'} + \lambda r^{N-1}\psi(u) = 0, \quad r \in [0, R], \quad U_n^{'}(0) = 0, \quad U_n(R) = 0,$$

with $U_n(0) = d_n$. Then $U_n(r) \to \infty$ uniformly with respect to r in compact subintervals of $[0, R)$

Proof

Since U_n is given by

$$U_n(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda_n \xi^{N-1}\psi(|U_n(\xi)|))d\xi\right] ds,$$

we obtain for $r > r_0$ (in corollary 4.1) that

$$U_n(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^{r_0} (\lambda_n \xi^{N-1}\psi(d_n\theta)d\xi\right] ds.$$

4.3.4 On The Principal Eigenvalue of The p - Laplacian

The consideration of the eigenvalue of the p - Laplacian allow for the proof of the fact that for the p - Laplacian; the eigenvalue problem

$$-\left[r^{N-1}\phi(u^{'})\right]^{'} = r^{N-1}\lambda\phi_p(u), \quad r \in (0, R), \quad u^{'}(0) = 0 = u(R) \tag{4.3.16}$$

has a unique principal eigenvalue with all eigenfunctions being a constant multiple of a given one ([1],[10]). To see this we observe that, because of the homogeneity of problem (4.3.16), constant multiple of the eigenfunctions are also eigenfunctions.Therefore, if $\lambda_1 < \lambda_2$ are principal eigenvalues of (4.3.16) with associated eigenfunctions u_1 and u_2. It follows from the hypothesis that $u_1^{'}(R) \neq 0 \neq u_2^{'}(R)$.

We next find constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha u_1(t) \leq u_2(t) \leq \beta u_1(t), 0 \leq t \leq R,$$

further, α may be chosen maximal and β minimal.This, on the other hand will imply that $\alpha = \beta$ and hence $u_2 = \alpha u_1$, which in turn implies that $\lambda_1 = \lambda_2$.

4.3.5 On The Higher Eigenvalues

Let us assume that $\psi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism of $\mathbb{R}$ with $\psi(0) = 0$. Also, we will require that ψ and ϕ satisfy the asymptotic homogeneity conditions:

$$\lim_{s\to 0}\frac{\phi(\sigma s)}{\psi s} = \sigma^{p-1}, \forall \ \ \sigma \in \mathbb{R}_+,$$

for some $p > 1$ and

$$\lim_{s\to\pm\infty}\frac{\phi(\sigma s)}{\psi s} = \sigma^{q-1}, \quad \forall \sigma \in \mathbb{R}_+,$$

for some $q > 1$. If the pair ϕ and ψ satisfies the asymptotic homogeneity conditions above, then the function ϕ satisfies both of these conditions with ψ replaced by ϕ and also ψ satisfies both of these conditions with ϕ replaced by ψ. These conditions are widely used in the work of ([19],[49],[20],[16]).

Finally, we re-affirm a well known fact that the eigenvalue problem (4.3.16) has a sequence $\{\lambda_n(p)\}_n$ of positive eigenvalues such that $\lambda_n(p) \to \infty$ as $n \to \infty$, and associated with each $\lambda_n(p)$ there is a one dimensional space spanned by a solution of (4.3.16) with exactly $n-1$ simple zeros in $(0, R)$ as in the work of ([11]).

4.4 On Initial Value Problems

Let us consider the problem

$$-\left[r^{N-1}\phi(u^{'})\right]^{'} = r^{N-1}\lambda\psi(u), \ in \ (0, R), u^{'}(0) = 0, u(R) = 0 \tag{4.4.1}$$

u is a solution to (4.4.1) if and only if u is a fixed point of the completely continuous operator

$T^{\lambda}_{\phi\psi} : C[0, R] \longrightarrow C[0, R]$ defined by

$$T^{\lambda}_{\phi\psi}(u)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s (\lambda\xi^{N-1}\psi(u(\xi))d\xi\right] ds. \tag{4.4.2}$$

In this section, we want to prove some results for the initial value problem associated with perturbation of the problem (4.4.1) i.e.,

$$-\left[r^{N-1}\phi(u^{'})\right]^{'} = r^{N-1}\lambda\psi(u) + r^{N-1}f(r, u, \lambda), in \quad (0, R), \quad u^{'}(0) = 0, \quad u(0) = d. \tag{4.4.3}$$

We shall assume throughout that $uf(r,u,\lambda) \geq 0$

Proposition

Suppose that $f(r,u,\lambda) = \circ(|u|)$ near zero, uniformly for r and λ in bounded intervals.Then the only solution to the problem

$$-[r^{N-1}\phi(u^{'})]^{'} = r^{N-1}\lambda\psi(u) + r^{N-1}f(r,u,\lambda), in(0,R), u^{'}(r_0) = 0, u(r_0) = 0,$$

with $r_0 \geq 0$ is the trivial solution $u = 0$, $\circ$ is a completely continuous operator in $\mathbb{R}$

Proof

Suppose u is a solution such that $u \neq 0$ in the interval $[r_0, r_0 + \delta)$, for some $\delta > 0$. Then integrating the equation from r_0 to $r \in [r_0, r_0 + \delta]$ as follows:

Given the problem

$$-\left[r^{N-1}\phi(u^{'})\right]^{'} = r^{N-1}\lambda\psi(u) + r^{N-1}f(r,u,\lambda)$$

$$-\int_{r_0}^{r}[\xi^{N-1}\phi(u^{'}(\xi))]^{'}d\xi = \int_{r_0}^{r}\left[\xi^{N-1}\lambda\psi(u(\xi)) + \xi^{N-1}f(\xi,u,\lambda)\right]d\xi$$

$$-[\xi^{N-1}\phi(u^{'}(\xi))]|_{r_0}^{r}\int_{r_0}^{r}\left[\xi^{N-1}\lambda\psi(u(\xi)) + \xi^{N-1}f(\xi,u\lambda)\right]d\xi$$

$$-r^{N-1}\phi(u^{'}(r)) + r_0^{N-1}\phi(u^{'}(r_0)) = \int_{r_0}^{r}\left[\xi^{N-1}\lambda\psi(u(\xi)) + \xi^{N-1}f(\xi,u,\lambda)\right]d\xi$$

But $u^{'}(r_0) = 0$

$$\therefore -r^{N-1}\phi(u^{'}(r)) = \int_{r_0}^{r}\left[\xi^{N-1}\lambda\psi(u(\xi)) + \xi^{N-1}f(\xi,u,\lambda)\right]d\xi,$$

$$-u^{'}(r) = \phi^{-1}\frac{1}{r^{N-1}}\int_{r_0}^{r}\left[\xi^{N-1}\lambda\psi(u(\xi)) + \xi^{N-1}f(\xi,u,\lambda)\right]d\xi$$

integrating both sides again,we have

$$-u(r) = \int_{r_0}^{r}\left[\phi^{-1}\frac{1}{s^{N-1}}\int_{r_0}^{s}\left[\xi^{N-1}\lambda\psi(u(\xi)) + \xi^{N-1}f(\xi,u,\lambda)\right]d\xi\right]ds$$

$$-u(r) = \int_{r_0}^{r}\left[\phi^{-1}\left(\frac{1}{s^{N-1}}\int_{r_0}^{s}\left[\xi^{N-1}\lambda\psi(u(\xi)) + \xi^{N-1}f(\xi,u\lambda)\right]d\xi\right)\right]ds.$$

Because of the assumption on f, there exist $t > 0$ such that for δ small
$|u(r)| \leq \int_{r_0}^{r} \left[\phi^{-1}\left(\frac{1}{s^{N-1}} \int_{r_0}^{s} \left[\xi^{N-1}\lambda\psi(u(\xi)) + t\psi(|u(\xi)|_\delta)\right] d\xi\right)\right] ds,$
where $|u(\xi)|_\delta$ denote the sup norm of u in $[r_0, r_0 + \delta]$. Hence

$$|u(\xi)|_\delta \leq \delta\phi^{-1}[(\lambda + t)\psi(|u(\xi)|_\delta)\delta],$$

and

$$\phi\left(\frac{|u|_\delta}{\delta}\right) \leq k\delta\psi(|u|_\delta),$$

where $k = (\lambda + t)$ independent of δ and

$$\frac{\phi\left(\frac{|u|_\delta}{\delta}\right)}{\phi(|u|_\delta)} \leq \frac{k\delta\psi(|u|_\delta)}{\phi(|u|_\delta)} \tag{4.4.4}$$

For δ small we obtain a contradiction, since the left hand side of the inequality (4.4.4) exceeds 1. A similar argument applies for an interval of the form $[r_0 - \delta, r_0]$, in case $r_0 > 0$.

4.5 Problems Of Annular Domain

In this section we consider the nonlinear differential equations of the form

$$\left[\phi(u^{'})\right]^{'} + \frac{N-1}{r}\phi(u^{'}) + f(u) = 0,\ 0 < r_1 < r < r_2,\ u = 0,\ r \in \{r_1, r_2\}, \tag{4.5.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism and which satisfies:

$$\forall \quad c > 0, \exists \quad A_c > 0,$$

such that

$$A_c\phi(u) \leq \phi(cu), u \in \mathbb{R}^+, \tag{4.5.2}$$

where

$$\lim_{c\to\infty} A_c = \infty. \tag{4.5.3}$$

Note that the above assumption immediately implies that $\exists\ B_c > 0$ such that

$$\phi(cu) \leq B_c\phi(u),\ u \in \mathbb{R}^+$$

with

$$\lim_{c\to 0} B_c = 0.$$

Since the only property of the function $\dfrac{N-1}{r}$ we shall use here is its continuity, we consider the more general problem

$$\left[\phi(u^{'})\right]^{'} + b(r)\phi(u^{'}) + f(u) = 0, 0 < r_1 < r < r_2, u = 0, r \in \{r_1, r_2\}, \tag{4.5.4}$$

where $b[r_1, r_2] \to \mathbb{R}$ is a continuous function.The nonlinear term f will be assumed to be continuous and satisfy

$$\lim_{u\to 0} \frac{f(u)}{\phi(u)} \leq 0 \tag{4.5.5}$$

and to grow superlinearly (with respect to ϕ) near infinity, i.e

$$\lim_{u\to \infty} \frac{f(u)}{\phi(u)} = \infty \tag{4.5.6}$$

i.e. f grows superlinearly with respect to ϕ near zero and infinity respectively.For such problems we shall establish that (4.5.4) always has a positive solution defined for any interval $[r_1, r_2]$.The result obtained may be viewed as extension of results to p - Laplacian like equations.

4.5.1 Fixed Point Formulation

Letting

$$P(r) = e^{\int_{r_1}^{r_2} b},$$

we may rewrite problem (4.5.4) equivalently as

$$\left[p\phi(u^{'})\right]^{'} + pf(u) = 0, 0 < r_1 < r < r_2, u = 0, r \in \{r_1, r_2\}. \tag{4.5.7}$$

We shall now establish the existence of solution of (4.5.7), hence (4.5.4) by proving the existence of fixed points of a completely continuous operator F,

$$F : c[r_1, r_2] = E \longrightarrow E,$$

where the norm of E is given by $\| u \| = max_r \in [r_1, r_2]|u(r)|$. The operator F is defined by the following lemma.

Lemma 4.1

Let ϕ be an odd increasing homeomorphism on $\mathbb{R}$ and satisfies condition (4.5.2)and (4.5.3) and C a non negative constant. Then for each $v \in E$ the problem

$$\left[p\phi(u^{'})\right]^{'} - cp\phi(u) = pv, 0 < r_1 < r < r_2,\ u = 0,\ r \in \{r_1, r_2\} \tag{4.5.8}$$

has a unique solution

$$u = T(v),$$

and the operator $T : E \to E$ is completely continuous.

Proof :

For each $\omega \in E$ let $u = B(\omega)$ be the unique solution of

$$\left[p\phi(u^{'})\right]^{'} - cp\phi(u) = pv, 0 < r_1 < r < r_2, u = 0, r \in \{r_1, r_2\},$$

i.e u is given by

$$u(r) = \int_{r_1}^{r} \phi^{-1}\left[\frac{1}{p}\left(q - \int_{r_1}^{s} p(c\phi(\omega)) + v\right)\right] ds$$

by integrating (4.5.8) where q is the unique number for which $u(r_2) = 0$.

Also from this follows that B is a completely continuous mapping. Now using the continuation theorem of Leray-Schauder to show that the operator B has a fixed point u, i.e. that

$$p\left[\phi(u^{'})\right]^{'} - cp\phi(\omega) = pv, 0 < r_1 < r < r_2, u = 0, r \in \{r_1, r_2\} \tag{4.5.9}$$

has a solution. We then define the operator T by

$$T(v) = u,$$

where u is the solution of (4.5.8). To accomplish what has been said, let $u \in E$ and $\lambda \in (0, 1)$ be such that

$$u = \lambda B(u).$$

Then

$$\left[p\phi\left(\frac{u^{'}}{\lambda}\right)\right]^{'} - cp\phi(u) = pv, 0 < r_1 < r < r_2, u = 0, r \in \{r_1, r_2\},$$

multiply the above equation by u and integrate i.e

$$\left[p\phi\left(\frac{u^{'}}{\lambda}\right)\right]^{'} u - ucp\phi(u) = pvu$$

Now integrate the first term by parts to obtain

$$p\phi\left(\frac{u^{'}}{\lambda}\right)u - \int_{r_1}^{r_2} p\phi\left(\frac{u^{'}}{\lambda}\right)u^{'} - \int_{r_1}^{r_2} cp\phi(u)u = \int_{r_1}^{r_2} pvu$$

but $u(r) = 0$

$$\therefore - \int_{r_1}^{r_2} p\phi\left(\frac{u^{'}}{\lambda}\right)u^{'} - \int_{r_1}^{r_2} cp\phi(u)u = \int_{r_1}^{r_2} pvu$$

$$\Longrightarrow \int_{r_1}^{r_2} p\phi\left(\frac{u^{'}}{\lambda}\right)u^{'} + \int_{r_1}^{r_2} cp\phi(u)u = -\int_{r_1}^{r_2} pvu$$

on the other hand, since ϕ is an increasing homeomorphism , for each $t > 0$ there exists a constant c_t such that

$$|x| \leq t\phi(x)x + c_t, x \in \mathbb{R}.$$

Thus we obtain

$$|\int_{r_1}^{r_2} pvu| \;\leq\; \| v \| \int_{r_1}^{r_2} p|u| \leq \| v \| \int_{r_1}^{r_2} (pt\phi(u)u + c_t),$$

(by definition of $\| \cdot \|$)

$$= \| v \| \int_{r_1}^{r_2} pt\phi(u)u + \| v \| \int_{r_1}^{r_2} c_t$$

$$= \| v \| \left[t\int_{r_1}^{r_2} p\phi(u)u + (c_t r_2 - c_t r_1)\right]$$

$$\Longrightarrow |\int_{r_1}^{r_2} pvu| \;\leq\; \| v \| \int_{r_1}^{r_2} p|u| \leq \| v \| \left[t\int_{r_1}^{r_2} p\phi(u)u + (c_t r_2 - c_t r_1)\right],$$

and choosing t appropriately

$$|\int_{r_1}^{r_2} pvu| \leq \frac{1}{2}c\int_{r_1}^{r_2} p\phi(u)u + c_1,$$

where $c_1 = c_t\,(r_2 - r_1)$ is a constant. Therefore we obtain

$$\int_{r_1}^{r_2} p\phi\left(\frac{u'}{\lambda}\right) u' \leq c_2,$$

for a constant c_2. Hence

$$\int_{r_1}^{r_2} |u'| \leq c_3,$$

and therefore

$$\| u \| \leq c_4.$$

Further

$$\| \left(p\phi\left(\frac{u'}{\lambda}\right)\right)' \|_{l^1} \leq c_5$$

since there exists r_0 such that $u'(r_0) = 0$, we obtain from the later inequality that

$$|p\phi\left(\frac{u'}{\lambda}\right)| \leq c_6$$

and hence

$$\| u' \| \leq c_7,$$

where $c_1, c_2, c_3, ..., c_7$ are constants independent of λ. B is completely continuous and we conclude that B has a fixed point. If u_1 and u_2 are fixed points,one immediately obtains that

$$\int_{r_1}^{r_2} p\phi(u_1' - u_2')(u_1' - u_2') + \int_{r_1}^{r_2} cp\phi(u_1 - u_2)(u_1 - u_2) = 0$$

and

$$\int_{r_1}^{r_2} p(\phi u_1' - \phi u_2')(u_1' - u_2') + \int_{r_1}^{r_2} cp(\phi u_1 - \phi u_2)(u_1 - u_2) = 0$$

and hence $u_1 = u_2$, since ϕ is increasing. Thus the operator given in the statement of the lemma is well defined.

4.6 Positone Problems

Here we are interested in the existence and multiplicities of positive solutions of the boundary value problem for a quasilinear differential equations.

$$\left[\phi(u')\right]' + \lambda f(t, u) = 0, a < t < b, u(a) = 0 = u(b) \tag{4.6.1}$$

with f continuous (but necessarily not locally Lipschitz continuous). We make the following assumptions:

ϕ is an odd increasing homeomorphism on $\mathbb{R}$,

$$\limsup_{x\to\infty} \frac{\phi(\sigma x)}{\phi(x)} < \infty, \forall \ \ \sigma > 0 \tag{4.6.2}$$

$f[a,b] \times [0,\infty) \to (0,\infty)$ is continuous and $\exists$ $[c,d] \subset (a,b), c < d$ such that

$$\lim_{u\to\infty} \frac{f(t,u)}{\phi(u)} = \infty, \text{uniformly for } t \in [c,d]. \tag{4.6.3}$$

In order to get our main result; we introduced the assumption (4.6.2) used in ([48],[38],[37],[2]) and was called σ-upper condition at $+\infty$. We want to re-emphasize that the condition is trivially satisfied if $\phi(x)$ is define as in the follow up of (4.3.5).

The main result in this section is:

Theorem 4.2

Assume that (4.6.2) and (4.6.3) hold. Then there exist a positive number λ^* such that the problem (4.6.1) has at least two positive solutions for $0 < \lambda < \lambda^*$, at least one for $\lambda = \lambda^*$ and none for $\lambda > \lambda^*$.

To prove the main theorem; we shall in addition to continuation methods use upper and lower solution methods ([39],[23],[8]). Since we are interested in non negative solutions we shall make the convention that $f(t,u) = f(t,0)$ if $u < 0$.

To do just this, we consider the following lemma:

Lemma 4.2

Let $v \in C^0[a,b]$ with $v \leq 0$ and let u satisfy

$$\left[\phi(u^{'})\right]^{'} = v, u(a) = 0 = u(b).$$

Then

$$u(t) \geq \| u \| p(t), t \in [a,b]$$

where

$p(t) = \dfrac{min(t-a, b-t)}{b-a}$.

Lemma 4.3

Suppose that $g : [a, b] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and there exists a positive number M and an interval $[a_1, a_2] \subset (a, b)$ such that

$$g(t, u) \geq M(\phi(u) + 1), t \in [a_1, a_2], u \geq 0$$

There exists a positive number $M_0 = M_0(\phi, a_1, b_1)$ such that the problem

$$\left[\phi(u^{'})\right]^{'} = -g(t, u), u(a) = 0 = u(b)$$

has no solution whenever $M \geq M_0$

Proof

Let u be a solution i.e. given the problem

$$\left[\phi(u^{'})\right]^{'} + g(t, u) = 0$$

integrating with respect to t from a to t, $t \in (a, b)$, we have

$$\phi\left(u^{'}(r)\right)|_a^t = -\int_a^t g(r, u)dr$$

$$\Longrightarrow \phi\left(u^{'}(t)\right) - \phi\left(u^{'}(a)\right) = -\int_a^t g(r, u)dr$$

put $\phi\left(u^{'}(a)\right) = c$, then

$$\phi\left(u^{'}(t)\right) = c - \int_a^t g(r, u)dr$$

$$\therefore u^{'}(t) = \phi^{-1}\left[c - \int_a^t g(r, u)dr\right]$$

$$u(t) = \int_a^t \phi^{-1}\left[c - \int_a^s g(r, u)dr\right] ds.$$

Let $\| u \| = u(t_0), t_0 \in [a, b]$. Then $u^{'}(t_0) = 0$ and hence

$$u(t) = \int_a^t \phi^{-1}\left[\int_a^{t_0} g(r, u)dr\right] ds$$

Now if $t_0 \geq \dfrac{a_1+b_1}{2}$, $[a_1, b_1] \subset [a, b]$ then

$$\| u \| \geq u(a_1) > \int_a^{a_1} \phi^{-1}\left[M \int_{a_1}^{\frac{a_1+b_1}{2}} (\phi(u)+1) \right]$$

$$> (a_1 - a)\phi^{-1}\left[M \frac{b_1 - a_1}{2} [\phi(\| u \| \delta) + 1] \right]$$

where $p(t) = \dfrac{min(t - a_1, b_1 - t)}{b_1 - a_1}$, and $\delta = \min_{a_1 \leq t \leq b_1} p(t)$.

This implies

$$\phi\left(\frac{\| u \|}{a_1 - a} \right) > M \frac{b_1 - a_1}{2} [\phi(\| u \| \delta) + 1]$$

If $t_0 \leq \dfrac{a_1+b_1}{2}$, then since

$u(t) = \int_t^b \phi^{-1}\left[\int_{t_0}^s g(r, u) dr \right] ds$,

we deduce

$$\phi\left(\frac{\| u \|}{b - b_1} \right) > \frac{M(b_1 - a_1)}{2} [\phi(\| u \| \delta) + 1]$$

combining the above, we obtain

$$\phi(\gamma \| u \|) > \frac{M(b_1 - a_1)}{2} [\phi(\| u \| \delta) + 1]$$

where $\gamma = \max\left(\dfrac{1}{b - b_1}, \dfrac{1}{a - a_1} \right)$

consequently

$$\| u \| \quad > \quad \frac{1}{\gamma} \phi^{-1}\left[\frac{M(b_1 - a_1)}{2} \right]$$

and

$$\frac{\phi(\gamma \| u \|)}{\phi(\delta \| u \|)} > \frac{M}{2}(b_1 - a_1)$$

which is a contradiction if M is sufficiently large and it follows that the problem in this lemma has no solution u satisfying

$$g(t, u(t)) = M[\phi(u(t)) + 1], t \in [a_1, a_2],$$

if $M \geq M_0$.

This contradiction also immediately imply that there exists a positive number $\bar{\lambda}$ such that problem (4.6.1) has no solution for $\lambda > \bar{\lambda}$.

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